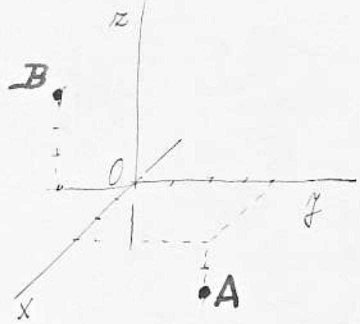
X. Základy anal. geom. v priestore

1§. Súradnicové systémy v priestore:

a)



Každý troj. reáln. č. priradíme práve 1 bod.

$A(x; y; z)$

Každému bodu v priest. priraď. jediné 1 bod

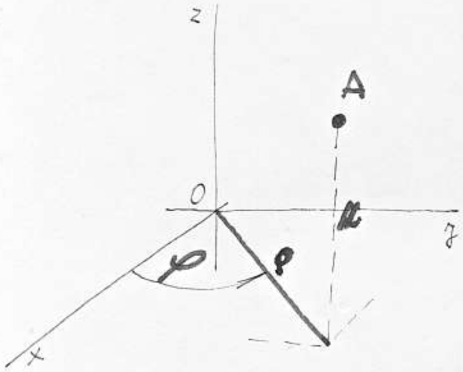
$A(3; 4; -2)$
$B(0; -2; 3)$ } pravouhlý s. s.

Cylindrické (válcové) súradnice

b)

$A(\varphi; \rho; z)$

$0 \leq \varphi < 2\pi$
$0 \leq \rho < +\infty$
$-\infty < z < +\infty$



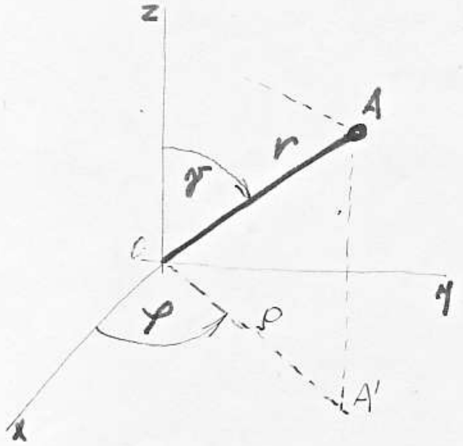
$x = \rho \cos \varphi$
$y = \rho \sin \varphi$
$z = z$

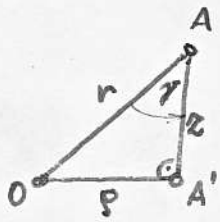
Sférické (guľové) súradnice

c)

$A(\varphi; \vartheta; r)$

$0 \leq \varphi < 2\pi$
$0 \leq \vartheta < \pi$
$0 \leq r < +\infty$

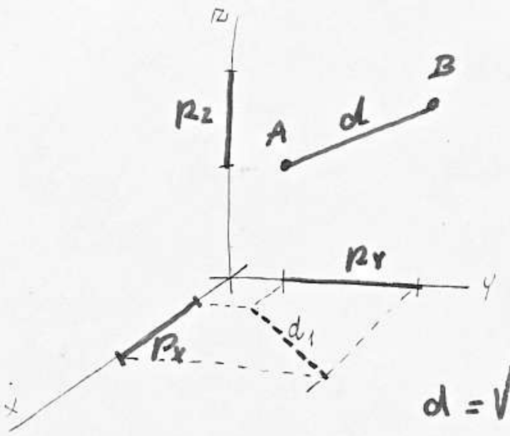




$\rho = r \cdot \sin \vartheta$

$x = r \sin \vartheta \cdot \cos \varphi$
$y = r \sin \vartheta \cdot \sin \varphi$
$z = r \cdot \cos \vartheta$

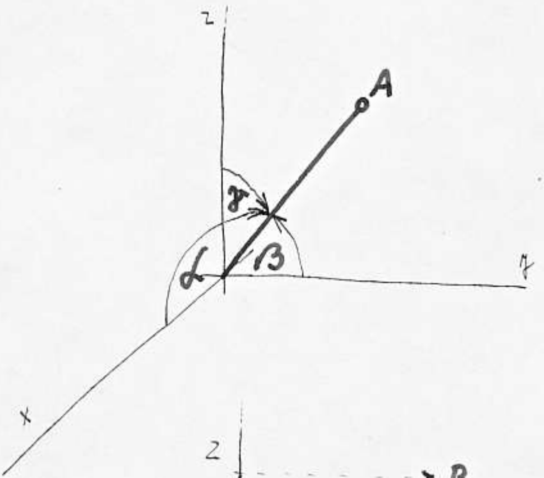
Priemety úsečky

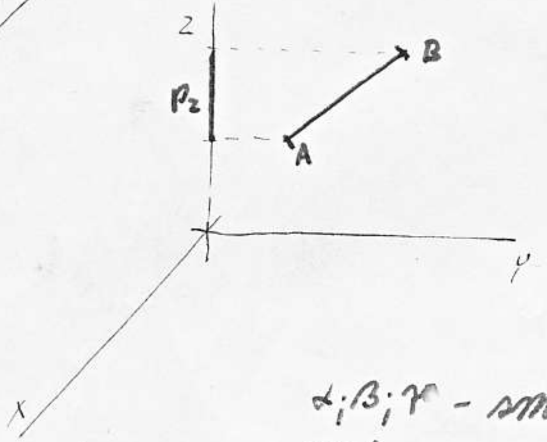


$A(x_1; y_1; z_1)$
$B(x_2; y_2; z_2)$ | $\overline{AB} = d$

$pr_x AB = p_x = x_2 - x_1$
$pr_y AB = p_y = y_2 - y_1$
$pr_z AB = p_z = z_2 - z_1$

$$d = \sqrt{p_x^2 + p_y^2 + p_z^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$





$p_z = AB \cos\gamma = d \cos\gamma$
$p_y = AB \cos\beta = d \cos\beta$
$p_x = AB \cdot \cos\alpha = d \cos\alpha$

$$\underbrace{p_x^2 + p_y^2 + p_z^2}_{d^2} = d^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma)$$

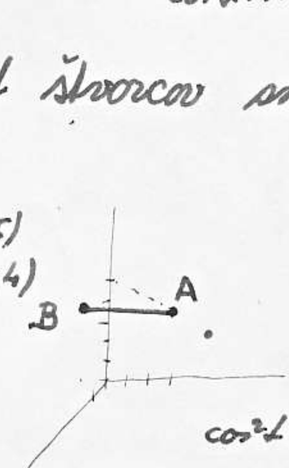
$\Downarrow$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$\alpha; \beta; \gamma$ – smerové uhly úsečky AB
$\cos\alpha \dots$ – smerové kosíny –"–

Súčet štvorcov smerových kosínov tej istej úsečky = 1

pr. : $A(1;3;5)$
$B(0;-1;4)$



$\cos\alpha = \frac{p_x}{d} = \frac{0-1}{3\sqrt{2}} = -\frac{1}{3\sqrt{2}}$ $\quad d = \sqrt{1+16+1} = \sqrt{18} = 3$

$\cos\beta = \frac{p_y}{d} = \frac{-1-3}{3\sqrt{2}} = -\frac{4}{3\sqrt{2}}$

$\cos\gamma = \frac{p_z}{d} = \frac{4-5}{3\sqrt{2}} = \frac{-1}{3\sqrt{2}}$

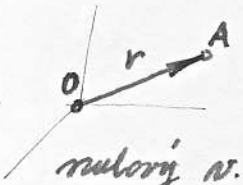
$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{1}{18} + \frac{16}{18} + \frac{1}{18} = 1$$

2 § Vektory a ich vlastnosti:

Z geom. hľadiska vektor defin. ako orientovanú úsečku.
označ.: $\bar{a} = \overline{AB}$

2. - v. považujeme za rovnaké ak majú rovnakú veľkosť a ten istý smer. : $\bar{a} = \bar{b}$

polohový v. (rádius v.) bodu A je taký v., ktorého poč. bod je poč. súr. syst.; koncový bod je bod A.



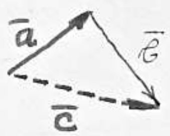
nulový v. : $\bar{0}$; je vektor, ktorého veľkosť je 0.
Je rovnobežný s každým v.

antivektor $\overline{AB}$ k vektoru $\overline{BA}$ je taký vektor, ktorý má tú istú veľkosť ako $\overline{BA}$, ale opačný smer.

Operácie s vektormi:

a. Sčitovanie vektorov:

$\bar{a} + \bar{b} = \bar{c}$



Dá sa dok. že pre sč. v. platia pravidlá:

1. $\bar{a} + \bar{b} = \bar{b} + \bar{a}$
2. $(\bar{a} + \bar{b}) + \bar{c} = \bar{a} + (\bar{b} + \bar{c})$
3. $\bar{a} + \bar{0} = \bar{a}$

b. Násobenie vektora číslom. ; $\bar{a}$; t ~ skalár (reáln. č)

$t.\bar{a}$ je vektor, ktorý má vlastnosti:

$t.\bar{a} = \bar{0}$; ak $t=0$
$t.\bar{a}$ má ten istý smer a je t násobkom vektora $\bar{a}$; ak $t>0$
$t.\bar{a}$ má opačný smer než $\bar{a}$; je $|t$ nás. $\bar{a}|$; ak $t<0$

Platia tieto pravidlá:

1.) $t(\bar{a}+\bar{b}) = t\bar{a}+t\bar{b}$

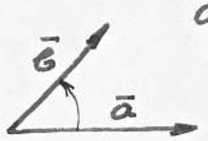
2.) $(t+s).\bar{a} = t\bar{a}+s\bar{a}$

3.) $t.(s\bar{a}) = (ts)\bar{a} = s(t\bar{a})$

4.) $t.\bar{0} = \bar{0}$

c.) Skalárny súčin 2 vektorov : $\bar{a}\,.\,\bar{b}$

$\bar{a}.\bar{b} = a.b.\cos\alpha$ ~ výsledok je skalár



1.) $\bar{a}.\bar{b} = \bar{b}.\bar{a}$

2.) $\bar{a}(\bar{b}+\bar{c}) = \bar{a}\bar{b}+\bar{a}\bar{c}$

3.) $t(\bar{a}.\bar{b}) = t.\bar{a}.\bar{b} = \bar{a}t\bar{b} = \bar{a}\bar{b}t$

Pozn.: ak $\bar{a}\perp\bar{b} \rightarrow \bar{a}.\bar{b} = ab\cos\frac{\pi}{2} = ab.0 = 0$

d.) Vektorový súčin 2 vektorov: - je vektor

$\bar{a}\times\bar{b} = \bar{c}$; $\bar{c}$ má tieto vlastnosti:

1.) je $\bar{c}\perp\bar{a}$

2.) $\bar{c}\perp\bar{b}$

$c = a.b.\sin\alpha$

$\bar{a}\times\bar{b} = -\bar{b}\times\bar{a}$

$\bar{a}\times(\bar{b}+\bar{c}) = \bar{a}\times\bar{b}+\bar{a}\times\bar{c}$

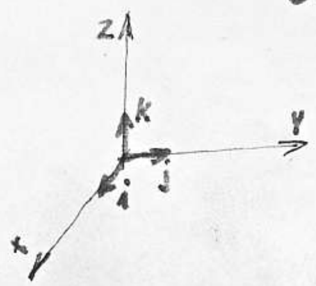
$t(\bar{a}\times\bar{b}) = (t\bar{a})\times\bar{b} = \bar{a}\times(t\bar{b})$

Pozn.: niekde vekt. súčin: $[\bar{a};\bar{b}]$

skal. súčin: $(\bar{a},\bar{b})$

Jednotkový v. - je v.; ktorého veľkosť = 1

$\bar{a}$ ...... $\bar{a}^{0}$; $\bar{a} = a.\bar{a}_0 \rightarrow \bar{a}_0 = \frac{\bar{a}}{a}$



(1) $\bar{a} = a_x i + a_y j + a_z k$

(2) $\bar{a} = (a_x, a_y, a_z)$

Operácie s vektormi vyjadrenými v zložkách.

a.) Sčítavanie vektorov:

$$\bar a = (a_x, a_y, a_z) \equiv i a_x + j a_y + k a_z$$
$$\bar b = (b_x, b_y, b_z) = i b_x + j b_y + k b_z$$

$$\bar a + \bar b = (a_x + b_x \,,\; a_y + b_y \,,\; a_z + b_z)$$
$$= i(a_x + b_x) + j(a_y + b_y) + k(a_z + b_z)$$

b.) Násobenie vektora číslom:

$$t.\bar a = i t a_x + j t a_y + k t a_z$$
$$= (t a_x \,,\; t a_y \,,\; t a_z)$$

c.) Skalárny súčin 2 vektorov:

$i.i = 1.1.1 = 1$ $\qquad i^2 = 1$ $\qquad i.j = 1.1.\cos\frac{\pi}{2} = 0$

$\qquad\qquad j^2 = 1$ $\qquad i.k = 0$

$\qquad\qquad k^2 = 1$ $\qquad j.k = 0$

$$\bar a.\bar b = (i a_x + j a_y + k a_z).(i b_x + j b_y + k b_z) =$$
$$= a_x b_x + a_y . b_y + a_z . b_z$$

Pozn.: $\left.\begin{aligned} \bar a.\bar a &= a_x^2 + a_y^2 + a_z^2 \\ (a.\bar a^0)(a.\bar a^0) &= a^2.1 = a^2 \end{aligned}\right\}\; a^2 = a_x^2 + a_y^2 + a_z^2$

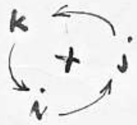
$$a = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

d.) Vektorový súčin v zložkách

14. IV. 1969.

$i \times i = \bar 0$ $\qquad d = 1.1.\sin 0 = 0$

$i \times i = 0$	$i \times j = k$	$j \times i = -k$
$j \times j = 0$	$j \times k = i$	$k \times j = -i$
$k \times k = 0$	$k \times i = j$	$i \times k = -j$



$$\overset{\longrightarrow\;+}{\underset{\longleftarrow\;-}{i\; j\; k\; i\; j\; k}}$$

$\bar a = i a_x + j a_y + k a_z$

$\bar b = i b_x + j b_y + k b_z$

$$\bar a \times \bar b = (i a_x + j a_y + k a_z) \times (i b_x + j b_y + k b_z) = k a_x b_y - j a_x b_z - k a_y b_x$$
$$+ i a_y b_z + j a_z b_x - i a_z b_y =$$
$$= i(a_y b_z - a_z b_y) - j(a_x b_z - a_z b_x) + k(a_x b_y - a_y b_x)$$

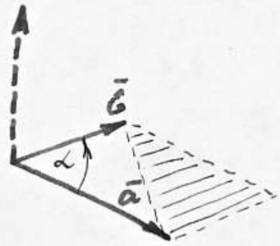
$$\bar a \times \bar b = i\begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - j\begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + k\begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

Príklad: $\bar{a}(1;-5;3)$
$\bar{b}(0;2;7)$

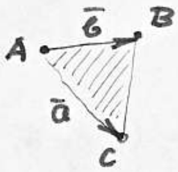
$$\bar{a}\times\bar{b} = \begin{vmatrix} i & j & k \\ 1 & -5 & 3 \\ 0 & 2 & 7 \end{vmatrix} = -35i + 2k - 6i - 7j \cancel{+2k} =$$

$$\bar{c} = (-41; -7; 2).$$

Geometrická aplik. vekt. súčinu.



$\bar{c} = \bar{a}\times\bar{b}$ $\qquad c = ab \sin\alpha$



$$P = \frac{1}{2}\, ab \sin\alpha$$

$$P = \frac{1}{2}\left|(\bar{a}\times\bar{b})\right|$$

Príkl.: Vyp. obs. Δ s vrcholmi:

$A(3;2;-1)$ $\quad B(0;4;3)$ $\quad C(1;-2;2)$

$\bar{a}(-3;2;4)$
$\bar{b}(-2;-4;3)$

$$P = \frac{1}{2}\left|\begin{vmatrix} i & j & k \\ -3 & 2 & 4 \\ -2 & -4 & 3 \end{vmatrix}\right| = \frac{1}{2}\left|6i - 12k - 8j + 4k + 16i + 9j\right| =$$

$$= \frac{1}{2}\left|22i + j - 8k\right| = \frac{1}{2}\sqrt{22^2 + 1^2 + 8^2} =$$

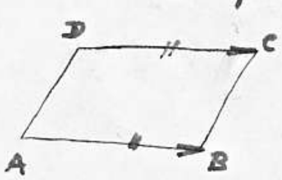
$$= \frac{1}{2}\sqrt{484 + 1 + 64};$$

e.) rovnobežnosť a kolmosť

2 vektory sú $\parallel$ práve vtedy, keď jeden z nich je číselným násobkom druhého.
(Ak sú nenulové – potom každý z nich je nás. druhého)

$\bar{b} = t\cdot\bar{a}$
$\bar{b} \parallel \bar{a}$

Príklad: vrcholy rovnobežníka: $A(-3;-2;0)$ $B(3;-3;1)$ $C(5;0;2)$. Nájsť vrchol $D(d_x;d_y;d_z)$



$\bar{a}(6;-1;1)$
$\bar{d}(5-d_x; 0-d_y; 2-d_z)$

$\bar{a} = 1\cdot\bar{d}$

$$\left.\begin{aligned} 5 - d_x &= 6 \\ 0 - d_y &= -1 \\ 2 - d_z &= 1 \end{aligned}\right\} \underbrace{d_x = -1 \quad d_y = 1 \quad d_z = 1}_{D(-1;1;1)}$$

Nech $\bar{a} \perp \bar{b} \Rightarrow \bar{a}\cdot\bar{b} = ab\cos\frac{\pi}{2} = 0 = \bar{a}\cdot\bar{b}$

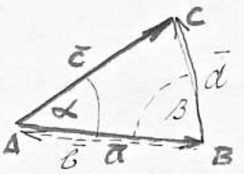
ak $\bar{a};\bar{b}$ sú nenulové vektory $\rightarrow \bar{a}\cdot\bar{b} = 0 \Leftrightarrow \bar{a}\perp\bar{b}$

f.) Uhol 2 vektorov

$$\bar{a}\cdot\bar{b} = ab\cdot\cos\varphi \quad\rightarrow\quad \cos\varphi = \frac{\bar{a}\cdot\bar{b}}{a\cdot b} = \frac{a_x b_x + a_y b_y + a_z b_z}{a\cdot b} =$$

$$= \frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{a_x^2+a_y^2+a_z^2}\cdot\sqrt{b_x^2+b_y^2+b_z^2}};$$

Pr.: vyp. uhly Δ: $A(2;-1;3)$ $B(1;1;1)$ $C(0;0;5)$



$\bar{a} = (-1;2;-2)$

$\bar{c} = (-2;1;2)$

$$\cos\alpha = \frac{2+2-4}{\sqrt{1+4+4}\sqrt{4+1+4}} = 0 \quad \alpha \Rightarrow \frac{\pi}{2}$$

$\bar{b}(1;-2;2)$

$\bar{d}(-1;-1;4)$

$$\cos\beta = \frac{-1+2+8}{\sqrt{1+4+4}\sqrt{1+1+16}} = \frac{9}{9\cdot\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$\beta = 45^\circ$

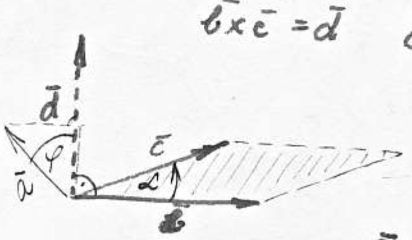
Zmiešaný (skalárno-vektorový) súčin 3 vektorov.

$\bar{a};\bar{b};\bar{c}$

$$\bar{a}\cdot(\bar{b}\times\bar{c}) = \bar{a}\cdot\bar{d} = ad\cos\varphi = abc\sin\alpha\cdot\cos\varphi$$

$\bar{b}\times\bar{c} = \bar{d}$ $\quad d = b\cdot c\cdot\sin\alpha$

$\bar{d}\perp\bar{b}$ $\quad \bar{d}\perp\bar{c}$.



Objem rovnobežnostena $(\bar{a};\bar{b};\bar{c})$

$$\bar{a}\cdot(\bar{b}\times\bar{c}) = \underbrace{b\cdot c\cdot\sin\alpha}_{\text{zákl.}}\cdot\underbrace{a\cdot\cos\varphi}_{\text{výška}}$$

$$\bar{a}\cdot(\bar{b}\times\bar{c}) = -\bar{a}(\bar{c}\times\bar{b})$$
$$(\bar{a}\times\bar{b})\cdot\bar{c} = \bar{a}(\bar{b}\times\bar{c})$$

$$(ia_x+ja_y+ka_z)\cdot\{(ib_x+jb_y+kb_z)\times(ic_x+jc_y+kc_z)\} =$$

$$= (ia_x+ja_y+ka_z)\cdot\{i(a_yb_z-a_zb_y) - j(a_xb_z-a_zb_x) + k(a_xb_y-a_yb_x)\} =$$

$$= a_x\cdot(a_yb_z-a_zb_y) - a_y(a_xb_z-a_zb_x) + a_z(a_xb_y-a_yb_x) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

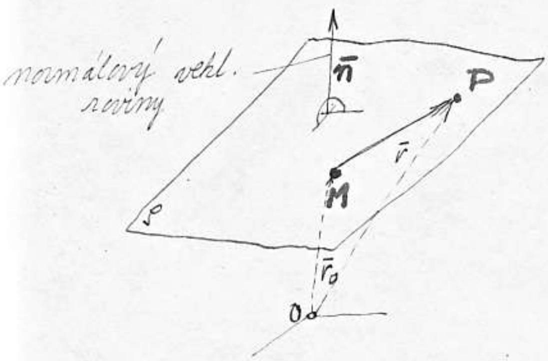
Príkl. : vyp. objem rovnobežnostena zostr. nad týmito smermi.

$\bar{a}(3;-2;0)$
$\bar{c}(0;2;0)$
$\bar{b}(-1;2;3)$

$$V=\begin{vmatrix} 3 & -2 & 0 \\ -1 & 2 & 3 \\ 0 & 2 & 0 \end{vmatrix} = -18$$

$$\underline{V = 18}$$

§3. Rovina



$M(x_0;y_0;z_0)$ ~ pevný bod

$\bar{n}\perp\rho \quad \bar{n}=(A;B;C)$

$P(x;y;z)$ ~ ľubov. bod roviny

$\overline{MP}$ leží v rovine

$\bar{r}_0 + \overline{MP} = \bar{r} \quad \rightarrow \quad \overline{MP} = \bar{r} - \bar{r}_0$

$\bar{n}\perp\rho \Rightarrow \bar{n}\perp\overline{MP} \Rightarrow \bar{n}\perp(\bar{r}-\bar{r}_0) \Leftrightarrow \bar{n}.(\bar{r}-\bar{r}_0)=0$

Vektorová rovnica roviny: $\bar{n}.(\bar{r}-r_0)=0 \quad (1)$

$$(i.A+j.B+k.C)\{i(x-x_0)+j(y-y_0)+k(z-z_0)\}=0$$

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0 \quad (2)$$

Pr. : nap. rovn. roviny prech: $M(3;5;-2)$ a je kolmá $\bar{n}(2;0;3)$

$2(x-3)+0(y-5)+3(z+2)=0$
$2x+3z=0$

Rovina je jednoznačne určená, ak je daný v. norm. a pevný bod roviny.

$$Ax+By+Cz\underbrace{-Ax_0-By_0-Cz_0}=0$$
$$Ax+By+Cz+D=0$$

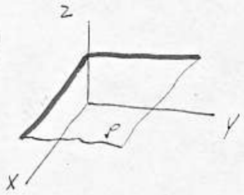
Zvláštne polohy roviny:

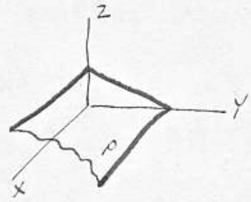
$Ax+By+Cz+D=0$

1.) $D=0$; $Ax+By+Cz=0$ ~ rovina prechádza počiatkom

2.) $A=0$; $By+Cz+D=0$ $\rho \parallel x$

$\bar{n}(0;B;C)$





$i=(1;0;0)$; $\bar{n}.i = 0.1 + B.0 + C.0 = 0$ $\cos\varphi = \dfrac{0.1+0.B+0.C}{\sqrt{1^2}\cdot\sqrt{B^2+C^2}} = 0$

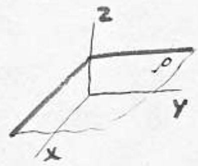
$\varphi = \dfrac{\pi}{2}$; $\underline{\underline{i \perp \bar{n}}}$

ak: $By+Cz=0$; – os x leží v rovine

3.) $A=B=0$; $Cz+D=0$ $\rho \parallel (x;y)$

$\bar{n}(0;0;C)$

$i.\bar{n}=0$
$j.\bar{n}=0$ $\Rightarrow \rho \parallel (x;y)$



Pozn: $x=0$ .... rovina $(y;z)$
$y=0$ ... $(x;z)$
$z=0$ ... $(x;y)$

4.) $A\neq 0$ $B\neq 0$ $C\neq 0$ $D\neq 0$

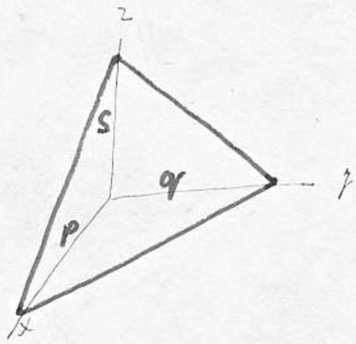
$$Ax+By+Cz=-D$$

$$\frac{Ax}{-D}+\frac{By}{-D}+\frac{Cz}{-D}=1$$

$$\frac{x}{-\frac{D}{A}}+\frac{y}{-\frac{D}{B}}+\frac{z}{-\frac{D}{C}}=1$$

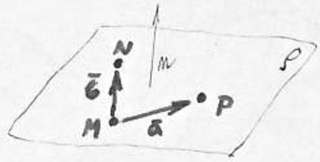
úsekový tvar roviny

$$\frac{x}{p}+\frac{y}{q}+\frac{z}{s}=1$$



Pr. nap. rovnicu roviny ; danej 3 bodmi.

$M(3;-2;0)$
$N(4;0;1)$
$P(-2;3;0)$



$\bar{m} = (\bar{a} \times \bar{b})$

$\bar{n} = \begin{vmatrix} i & j & k \\ -5 & 5 & -1 \\ 1 & 2 & 0 \end{vmatrix} = -10k - j - 5k + 2i$

$\bar{n} = (2;-1;-15)$

$$2(x-3) - 1(y+2) - 15(z-1) = 0$$
$$2x - y - 15z = 7$$

Pr. Zistite či roviny $2x - 3y + z = 4$
$4x - 6y + 2z = 3$ sú rovnobežné.

$n_1(2;-3;1)$
$n_2(4;-6;2)$

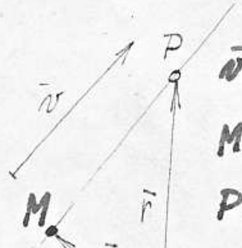
$$\cos\varphi = \frac{8+18+2}{\sqrt{4+9+1}\sqrt{16+36+4}} = \frac{28}{\sqrt{14}.\sqrt{56}} = \frac{28}{\sqrt{16.49}} = 1$$

$\varphi = 0$

Každá algebraická rovnica 1. st. o (premenných) neznámych $x;y;z$ je .. rovina.
Každá rov. v priest. je daná alg. rovnicou 1. st. o neznámych $x;y;z$.

16.V.1969.

§4. Priamka v priestore.



$\bar{v}(m;n;p)$ $\bar{r}_0(x_0;y_0;z_0)$ / v ~ smerový vektor priamky.
$M(x_0;y_0;z_0)$ $\bar{r}(x;y;z)$
$P(x;y;z)$ $\overline{MP} = \bar{r} - r_0 = t.\bar{v}$ / $\bar{v} \parallel \overline{MP}$

$\bar{r} = \bar{r}_0 + t\bar{v}$ ~ rovnica priamky

$$ix + jy + kz = ix_0 + jy_0 + kz_0 + itm + jtn + ktp$$
$$ix + jy + kz = i(x_0 + tm) + j.(y_0 + tn) + k(z_0 + tp)$$

$$\begin{aligned} x &= x_0 + tm \\ y &= y_0 + tn \\ z &= z_0 + tp \end{aligned} \quad (2)$$ parametrické rovnice priamky ; $(-\infty < t < +\infty)$

Pr. nap. par. rovn. priamky $\parallel$ s $\vec{v}(3;2;1)$ a $M(-2;0;4)$:

$$\begin{aligned} x &= \underbrace{-2}_{} + 3t \\ y &= 0 + 2t = 2t \\ z &= 4 + t \end{aligned}$$

súr. pevn. bodu

$p(2)$ $\rightarrow$

$$\left.\begin{aligned} t &= \frac{x-x_0}{m} \\ t &= \frac{y-y_0}{n} \\ t &= \frac{z-z_0}{p} \end{aligned}\right\} \quad \frac{x-x_0}{m} = \frac{y-y_0}{n} = \frac{z-z_0}{p}$$

~ základný tvar rovnice priamky.

Pr.: $\frac{x+2}{3} = \frac{y-0}{2} = \frac{z-4}{1}$

<u>Pozn.</u> Nap. rovn. priamky prech. $M;N$: $M(3;2;4)$; $N(-1;0;5)$

$$\left.\begin{aligned} &\vec{v} = (-4;-2;1) \\ &M(3;2;4) \end{aligned}\right\} \quad \frac{x-3}{-4} = \frac{y-2}{-2} = \frac{z-4}{1}$$

Teoreticky:

$$\left.\begin{aligned} &M(x_1;y_1;z_1) \\ &N(x_2;y_2;z_2) \end{aligned}\right\} \quad \frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_2}$$

§5. Vzť. medzi priamkou a rovinou.

$\rho \ldots \ Ax + By + Cz + D = 0$

$p \ldots \ x = x_0 + mt$
$\quad\quad\ y = y_0 + nt$
$\quad\quad\ z = z_0 + pt$

Spoločné body rov. a priamky (riešiť sústavu)

$$A(x_0+mt) + B(y_0+nt) + C(z_0+pt) + D = 0$$

$$t\underbrace{(Am + Bn + Cp)}_{u} = \underbrace{-D - Ax_0 - By_0 - Cz_0}_{V}$$

$$ut = V \qquad (3)$$

1, $u \neq 0 \Rightarrow t = \frac{V}{u}$ ~ ex. 1. riešenie pre t. - čo dosad. do rovníc. dostávame priesečík: $P(x_3;y_3;z_3)$

2, $u = 0$; $V = 0 \Rightarrow 0.t = 0$ ~ rieš. ex. pre každé t. - priamka leží v rovine

3, $u = 0$; $V \neq 0 \Rightarrow 0t \neq 0$ ~ nemá rieš. ~ priamka je $\parallel$ s rovinou.

Pr. Nájď pries. pr. s rovinou.

$$\frac{x-1}{2}=\frac{y}{3}=\frac{z+7}{1}$$
$$3x-2y+z+2=0$$

$$\left.\begin{array}{l} x-1=2t \rightarrow x=2t+1 \\ y=3t \rightarrow y=3t \\ z+7=t \rightarrow z=t-7 \end{array}\right.$$

$$3(1+2t)-2(3t)+(-7+t)+2=0 \qquad t=2 \qquad \left.\begin{array}{l} x=1+2\cdot 2 \\ y=3\cdot 2 \\ z=-7+2 \end{array}\right\} P(5;6;-5)$$

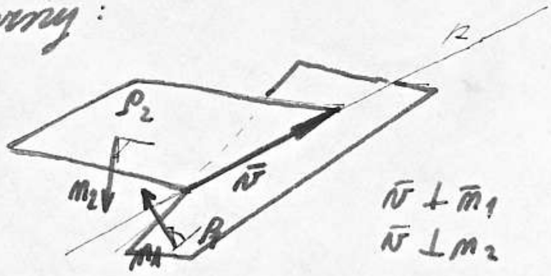
Skúška: $\underline{\frac{5-1}{2}=\frac{6}{3}=\frac{2}{1}}$; $3.5-2.6-5+2=0$

Majme 2 rôznobežné roviny:

$\rho_1 \ldots \; A_1x+B_1y+C_1z+D_1=0$

$\rho_2 \ldots \; A_2x+B_2y+C_2z+D_2=0$

$n_1=(A_1;B_1;C_1) \quad n_2=(A_2;B_2;C_1)$



$\bar{v}\perp\bar{n}_1$
$\bar{v}\perp n_2$

$$\left.\begin{array}{l}\bar{v}=\bar{n}_1\times\bar{n}_2 \\ M(x_0;y_0;z_0) \quad \text{(za 1 súr. v rovine si zvol. ľubov [0])}\end{array}\right\} \text{priamka}$$

Pr.: Nap. rovn. priamky:

$$\left.\begin{array}{l}3x+2y-z+4=0 \\ x+4y+2z=3\end{array}\right\} \left.\begin{array}{l}\bar{n}_1(3;2;-1) \\ \bar{n}_2(1;4;2)\end{array}\right\} \bar{v}=\begin{vmatrix} i & j & k \\ 3 & 2 & -1 \\ 1 & 4 & 2\end{vmatrix}=(8;-7;10)$$

$$\begin{array}{l}3x+2y-z=-4 \\ x+4y+2z=3\end{array} \quad M(\;,\;,0)$$

$$\begin{array}{l}3x+2y=-4 \\ x+4y=3 \\ \hline -10y=-13 \\ y=\frac{13}{10}\end{array} \qquad x=-\frac{11}{5} \qquad M\left(-\frac{11}{5};\frac{13}{10};0\right)$$

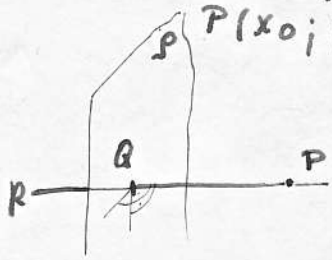
$$x=-\frac{11}{5}+8t$$
$$y=\frac{13}{10}-7t$$
$$z=0+10t$$

§ 6; Vzd. bodu od priamky; roviny

a.) vzd. bodu od roviny

$\rho \ldots \quad Ax + By + Cz + D = 0$

$P(x_0; y_0; z_0)$



$d = PQ$

$\vec{n}(A; B; C) \quad P(x_0; y_0; z_0)$

$$x = x_0 + At$$
$$y = y_0 + Bt$$
$$z = z_0 + Ct$$

$$A(x_0 + At) + B(y_0 + Bt) + C(z_0 + Ct) + D = 0$$
$$t(A^2 + B^2 + C^2) = -(Ax_0 + By_0 + Cz_0 + D)$$
$$t = -\frac{Ax_0 + By_0 + Cz_0 + D}{A^2 + B^2 + C^2}$$

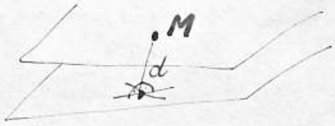
$$d = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$$

$$d = \sqrt{A^2t^2 + B^2t^2 + C^2t^2} = |t|\sqrt{A^2 + B^2 + C^2} \Rightarrow$$

(4) $$d = \left| \frac{Ax_0 + By_0 + Cz_0 + D}{A^2 + B^2 + C^2} \right| \cdot \sqrt{A^2 + B^2 + C^2} = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

Pr.: Vyp. vzd. medzi 2 rovnob. rovinami:

$$x + 2y + 3z - 8 = 0$$
$$x + 2y + 3z - 12 = 0$$



zvol. $M(8; 0; 0)$

$$d = \frac{|8 + 2 \cdot 0 + 3 \cdot 0 - 12|}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{4}{\sqrt{14}}$$

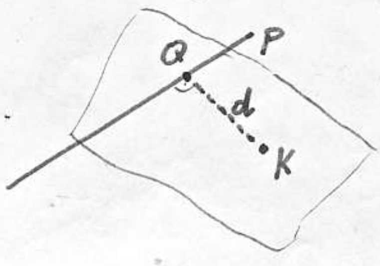
b.) vzd. bodu od priamky

Dané: p; K

Postup: 1.) bodom K preložíme rovinu kolmú na priamku p.

2.) nájd. priesečník p s rovinou : Q

3.) vyp. vzd. $d = \overline{KQ}$



Pr: vzd. $K(3;0;4)$ od pr. $x = 3+2t$
$y = 4-t$
$z = 2$

$\bar{n}_\rho = (2;-1;0)$ ≡ vektorom normály pre ρ.

$$2(x-3)-1(y-0)+0(z-4)=0$$

$2x-y-6=0$ ~ rovn. kolm. roviny

$$2(3+2t)-(4-t)-6=0$$

$5t = 4 \rightarrow t = \frac{4}{5}$

$$Q\left(\frac{23}{5};\frac{16}{5};2\right)$$

$$d = \sqrt{\left[\left(\frac{23}{5}\right)-3\right]^2+\left(\frac{16}{5}\right)^2+(4-2)^2} = \dots ;$$

§7. Uhol 2 rovín; 2 priamok; priamky a roviny.

a.) uhol 2 rovín:

$\rho_1 \dots A_1x+B_1y+C_1z+D_1=0$
$\rho_2 \dots A_2x+B_2y+C_2z+D_2=0$

Uhol 2 rovín def.; ako uhol ich norm. vektorov.

$$\cos\varphi = \pm\frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\sqrt{A_2^2+B_2^2+C_2^2}}$$

Ak $\rho_1 \perp \rho_2 \Rightarrow \varphi = \frac{\pi}{2} \Rightarrow \cos\varphi = 0 \Rightarrow A_1A_2+B_1B_2+C_1C_2=0$

podmienka kolmosti 2 rovín.

Ak $\rho_1 \parallel \rho_2 \Rightarrow \bar{n}_1 = t\bar{n}_2 \Rightarrow A_1 = tA_2;\ B_1 = tB_2;\ C_1 = tC_2$

$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$ podm. rovnob.

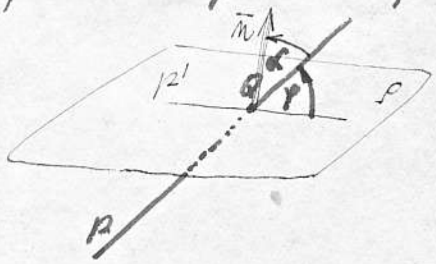
b.) uhol 2 priamok — uhly ich smer. vektorov.

$p_1 \cdots \bar{v}_1 = (m_1\ n_1\ p_1)$
$p_2 \cdots \bar{v}_2 = (m_2\ n_2\ p_2)$

$$\cos\varphi = \frac{|m_1 m_2 + n_1 n_2 + p_1 p_2|}{\sqrt{m_1^2+n_1^2+p_1^2}\cdot\sqrt{m_2^2+n_2^2+p_2^2}}$$

ak $p_1 \perp p_2 \Rightarrow m_1 m_2 + n_1 n_2 + p_1 p_2 = 0$
ak $p_1 \parallel p_2 \Rightarrow \frac{m_1}{m_2} = \frac{n_1}{n_2} = \frac{p_1}{p_2}$

c.) uhol priamky a roviny: $Ax + By + Cz + D = 0$; $x = \dots\ y = \dots\ z = \dots$



p' = priemet p do ρ

$\bar{n}(A;B;C)$
$\bar{v}(m;n;p)$

$\alpha + \varphi = \frac{\pi}{2}$

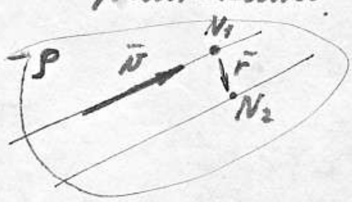
$$\cos\alpha = \frac{\bar{n}.\bar{v}}{n.v}$$

$$\cos\left(\frac{\pi}{2}-\varphi\right) = \sin\varphi = \frac{\bar{n}.\bar{v}}{n.v}$$

$$\sin\varphi = \frac{|mA + nB + pC|}{\sqrt{m^2+n^2+p^2}\cdot\sqrt{A^2+B^2+C^2}}$$

ak $p \parallel \rho \Rightarrow mA + nB + pC = 0$
ak $p \perp \rho \Rightarrow \frac{m}{A} = \frac{n}{B} = \frac{p}{C}$

pr. nap. rovn. roviny, ktorá prech. rovnobežnými priamkami



$$\frac{x-2}{4} = \frac{y+3}{-1} = \frac{z}{5}$$

$$\frac{x}{4} = \frac{y-5}{-1} = \frac{z+2}{5}$$

$N_1(2; -3; 0)$
$N_2(0; 5; -2)$

$\bar{v} = (4; -1; 5)$
$\bar{r} = (-2; 8; -2)$

$$\bar{n} = \bar{v}\times\bar{r} = \begin{vmatrix} i & j & k \\ 4 & -1 & 5 \\ -2 & 8 & -2 \end{vmatrix} = (-38, -2, 30)$$

$-38(x-2) - 2(y+3) + 30z = 0.$

$-38x - 2y + 30z + 70 = 0$ $\qquad$ $+19x + y - 15z - 35 = 0$

21. IV. 1969.

Plochy def. alg. rovn. 2° o premenných x, y, z nazývame plochami 2. stupňa. Všeob. tvar:

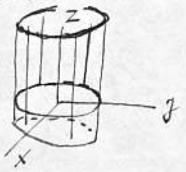
$$Ax^2+By^2+Cz^2+Dx+Ey+Fz+L=0$$
$$+\,Gxy+Hxz+Kyz$$

Aspoň 1 z $A, B, C, G, H, K \neq 0$

Špec. prípady:

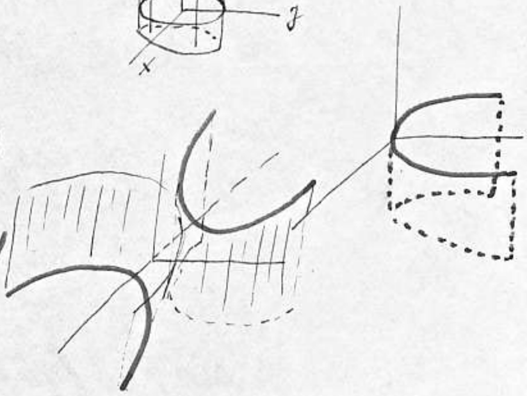
a., valcové plochy: ak rovn. (1) obs. len 2 premenné; potom nám predst. valcovú plochu, ktorej tvoriace priamky sú $\parallel$ s tou so súr. osí, ktorej prem. v dan. rovnici nevystupuje.

1., elypt. valc. plocha: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



2., parabolická valc. plocha: $\frac{x^2}{a^2}=2py$

3., hyperbolická v. pl.: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$



b., kužeľové plochy: ak rovn. (1) obsahuje vš. 3 premenné a dá sa transformovať na tvar:

$Ax^2+By^2+Cz^2=0$ (2) potom predst. kuž. plochu.

Ak leží na ploche $P(x;y;z)$, leží na nej aj $R(kx;ky;kz)$ ~ na ploche ležia priamky – priamková plocha:

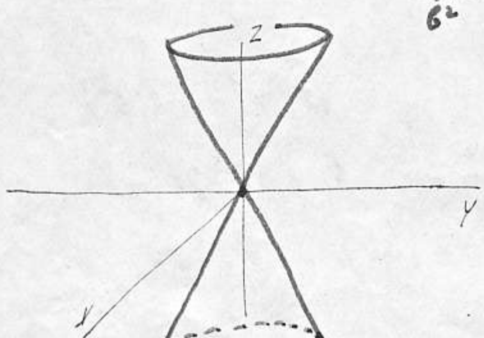
$x_1=kx$; $\frac{x_1}{x}=k$; $\frac{x_1}{x}=\frac{y_1}{y}=\frac{z_1}{z}$;

1., elipt. kužeľová plocha.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$

rovn. zy : $x=0$

$$\frac{y^2}{b^2}-\frac{z^2}{c^2}=0=\left(\frac{y}{b}+\frac{z}{c}\right)\left(\frac{y}{b}-\frac{z}{c}\right)=0 < \begin{matrix} y=\frac{b}{c}z \\ y=-\frac{b}{c}z \end{matrix}$$ rovn. zy



rov $\parallel$ (xy) $z=k$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=k$ ~ elipsa c.) Guľa – množ. bodu priest.; rovn. vzdial. od pevn. bodu; ...

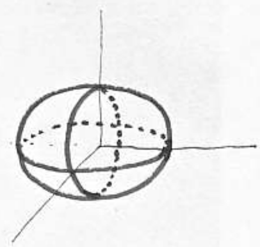
$$x^2+y^2+z^2=r^2$$

d.) Elipsoid – $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ~ základný; kanon. tvar.

Rezy: $(xy) \to z=0$ $\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$(xz) \to y=0$ $\quad \frac{x^2}{a^2}+\frac{z^2}{c^2}=1$

$(yz) \to x=0$ $\quad \frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

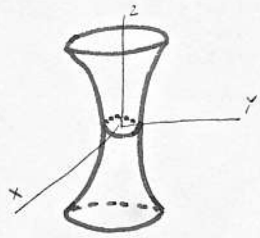


e.) Jednodielny hyperboloid: $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ (1 znamienko –)

Rezy: $(z=0)$ $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ – elipsa

$(y=0)$ $\frac{x^2}{a^2}-\frac{z^2}{c^2}=1$ – hyperb.

$(x=0)$ $\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ – hyperb.



os z nepretína plochu: $-\frac{z^2}{c^2}=1 \Rightarrow z^2=-c^2$

$y^2=b^2 \Rightarrow y_{1,2}=\pm b$

$x^2=a^2 \Rightarrow x_{1,2}=\pm a$

$x=a$ $\quad 1+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$; $\frac{y^2}{b^2}-\frac{z^2}{c^2}=0 \Rightarrow \left.\begin{array}{l} y=\frac{b}{c}z \\ y=-\frac{b}{c}z \end{array}\right\}$ dvojica priamok

analogicky pre $y=0$.

~ j. hyperb. je priamk. plocha

f.) dvojdielny hyperboloid:

$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ (2 znam. –)

$(x;y)$; $z=0$ $\quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ ~ hyp.

$(x;z)$; $y=0$ $\quad \frac{x^2}{a^2}-\frac{z^2}{c^2}=1$ ~ hyp.

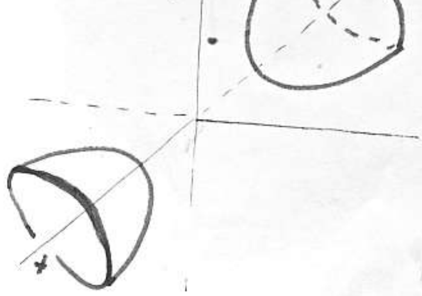
$(z;y)$; $x=0$ $\quad -\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ $\quad /\cdot(-1)$

$\frac{y^2}{b^2}+\frac{z^2}{c^2}=-1$ ~ rez neex.

$x=y=0$ $\quad -z^2=c^2$ – neex. priesečník

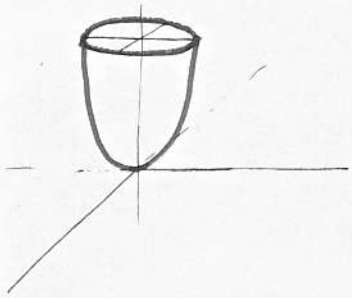
$x=z=0$ $\quad -y^2=b^2$ – neex. –"–

$y=z=0$ $\quad x^2=a^2$ – $x_{1,2}=\pm a$



g. *Eliptický paraboloid*

$$\frac{x^2}{p} + \frac{y^2}{q} = 2rz \qquad / \; p>0 \,;\; q>0 \,;\; r>0$$

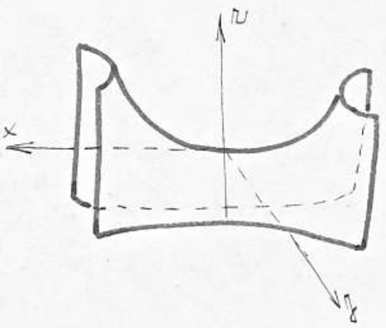


$(x;y)$	$z=0$	$\frac{x^2}{p} + \frac{y^2}{q} = 0$	$\Rightarrow O(0;0;0)$
pre $\parallel$ (xy)	$z=k$	$\frac{x^2}{p} + \frac{y^2}{q} = k$	$\Rightarrow$ elipsa
$(x;z)$	$y=0$	$\frac{x^2}{p} = 2rz$	$\Rightarrow$ parab.
$(y;z)$	$x=0$	$\frac{y^2}{q} = 2rz$	$\Rightarrow$ parab.

h. *Hyperbolický paraboloid*

$$\frac{x^2}{p} - \frac{y^2}{q} = 2rz \qquad p>0 \,;\, q>0 \,;\, r>0 \,;$$

$(x;y)$ $z=0$ $\quad \frac{x^2}{p} - \frac{y^2}{q} = 0 \Rightarrow \begin{cases} y = \sqrt{\frac{q}{p}}\,x \\ y = -\sqrt{\frac{q}{p}}\,x \end{cases}$



$(x;z)$ $y=0$ $\quad \frac{x^2}{p} = 2rz \Rightarrow$ parab.

$(y;z)$ $x=0$ $\quad \frac{-y^2}{q} = 2rz \Rightarrow$ parab.

$x=y=0 \Rightarrow z=0 \rightarrow O(0;0;0)$

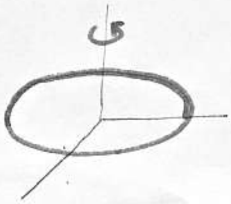
i. *rot. plochy* . plocha ktorú opíše krivka k rot. okolo priamky p naz. sa rot. plochou s osou rotácie p. Pretože každý bod kr. k pri rot. opisuje kružnicu; nachádza sa v rov. kolmej na priamku p. Ak rovnica pl. má tvar:

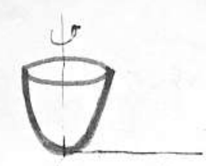
$F(x^2+y^2;z)=0$ ... os rot.: $\underline{z}$

$F(x;y^2+z^2)=0$ ... os rot.: $\underline{x}$

$F(x^2+z^2;y)=0$ ... os rot.: $\underline{y}$

Napr.: $\frac{x^2+y^2}{a^2} + \frac{z^2}{c^2} = 1$ ~ rot. elipsoid ; os rot.: $\underline{z}$





$\frac{x^2+y^2}{p} = 2rz$ ~ rot. paraboloid

Pr.: Najdite prieč. priamky $x = 4+2t$, $y=-6-3t$, $z=-2-2t$ s elipsoidom $\frac{x^2}{16}+\frac{y^2}{12}+\frac{z^2}{4}=1$

$3x^2+4y^2+12z^2=48$

$3(4+2t)^2+4(-6-3t)^2+12(-2-2t)^2=48$

$48+48t+12t^2+144+144t+36t^2+\dots=0 < \begin{matrix} t_1=-1 \\ t_2=-2 \end{matrix}$ | $Q(0,0,2)$, $P(2;-3;0)$

Pr.: Dokážte že na elipt. paraboloide neležia priamky

$\frac{x^2}{p}+\frac{y^2}{q}=2rz \qquad (p;q;r)>0$

$qx^2+py^2=2rqpz$

$qx^2+py^2=Rz$

$x=x_0+mt$
$y=y_0+nt$
$z=z_0+p_1t$

$q(x_0+mt)^2+p(y_0+nt)^2=R(z_0+p_1t)$

$qx_0^2+2mtxq+qm^2t^2+p(y^2+2nty+n^2t^2)=R(z_0+p_1t)$

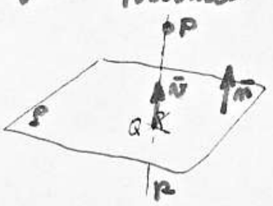
$t^2(m^2q+n^2p)+t(2mx_0q+2ny_0p+Rp_1)+qx_0^2+py_0^2-Rz_0=0$

$At^2+Bt+C=0$

keď priamka leží v ploche; rovnica má mať rieš. pre ľubov. t. $\Rightarrow A=B=C=0$

$m^2q+n^2p=0 \quad \sim$ spor $\Rightarrow$ priamka neleží na ploche.

Pr. Najd. priemet bodu $P(3;0;4)$ na rovinu $\varrho \dots x-2y+13=0$



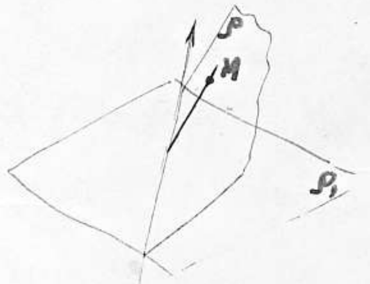
$\vec{n}(1;-2;0)=\vec{v}$

$p \begin{cases} x=3+t \\ y=0-2t \\ z=4+0 \end{cases}$

$3+t+4t+13=0$
$5t=-16$
$t=-\frac{16}{5}$

$Q\left(-\frac{1}{5};+\frac{32}{5};4\right)$

Pr. Nap. rovn. roviny, ktorá prech. bodom $M(2;0;1)$ a je kolmá na $\varrho_1 \dots x+2y+z-1=0$

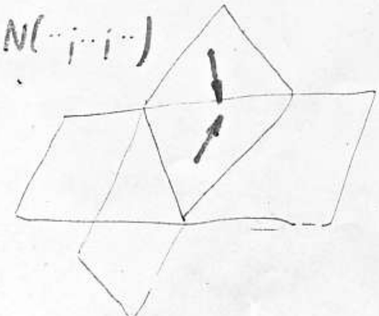


$\vec{n}_1(1;2;1)$
$\vec{n}(A;B;C)$

$1\cdot A+2B+1C=0$

$\boxed{\begin{matrix} A+2B+C=0 \\ 2A+0B+C+D=0 \end{matrix}}$

$+ N(\cdot;\cdot;\cdot)$



$\vec{m}_1(1;2;1)$

$\vec{v}=(1;-5;3)$

$\vec{n}=\vec{m}_1\times\vec{v}$

23. IV. 1969.

FUNKCIE dvoch a viac premenných.

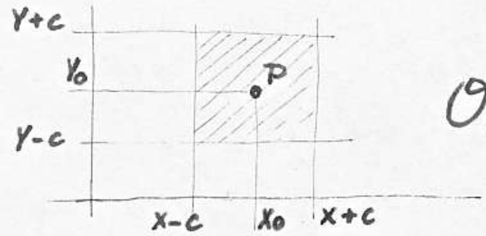
§1. Množiny

E_2 – budeme rozumieť rovinu; množinu bodov $P(x;y)$

Štvorcové okolie bodu $P_0(x_0;y_0)$.

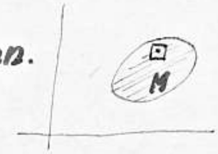
Nech $c>0$; pod štv. ok. bodu P rozumieme body roviny ktoré splňujú tieto podmienky: $x_0-c<x<x_0+c$; $|x-x_0|<c$

$y_0-c<y<y_0+c$; $|y-y_0|<c$



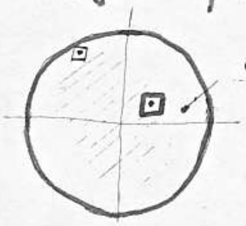
$O(P_0)$

E_2; M - mn.



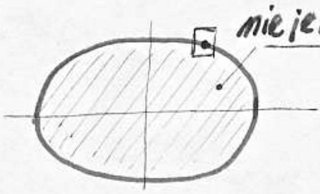
Množina M je otvorená ak pre každý bod P z mn. M ex. $O(P)$, že $O(P)\subset M$

Pr.: $x^2+y^2<r^2$; r = konšt.



otv. množ.

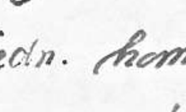
Pr.: $\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1$



nie je otv.

$M\subset E_2$ $M\neq\emptyset$ M' - ozn. súhrn bodov z E_2; takých ktoré nepatria do M. Potom M' je komplement množiny M. (doplnok)

$M''=(M')'=M$

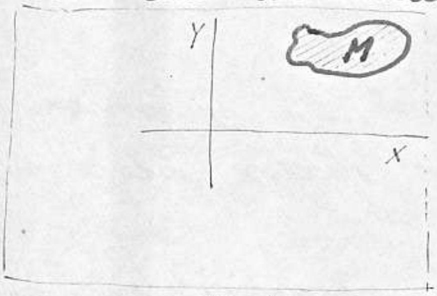


$M'\cup M=E_2$ zjedn. kompl. a množ. – dáva celú rovinu

množina M je uzavretá ak jej komplement je množina otvorená.

Pozn.: Celá rovina E_2 je množina súčasne otvorená aj uzavretá.

Množinu M nazývame ohraničenou ak ex. také okolie bodu $O(0,0)$, že celá množina padne do toho okolia.

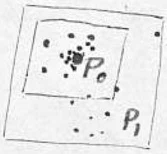


Majme postupnosť bodov roviny $P_1;\ P_2;\ P_3;\ \dots\ P_n;\ \dots P_0$

$P_1(x_1; y_1)$ $\quad P_2(x_2; y_2)$ $\quad \dots \quad P_n(x_n; y_n) \dots P_0(x_0; y_0)$.

$P_1, P_2, P_3 \dots \qquad (1)$

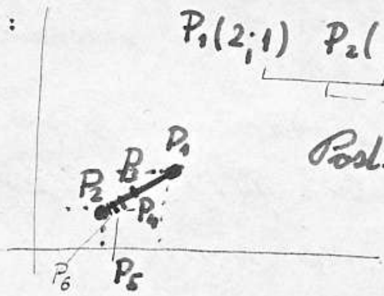
Hovoríme že post. (1) konverguje k bodu P_0, ak v ľub. okolí bodu P_0 ležia skoro vš. členy postupnosti (1).



$$P_n \to P_0 \iff \begin{cases} x_n \to x_0 \\ y_n \to y_0 \end{cases}$$

$$\lim_{n \to \infty} P_n = P_0$$

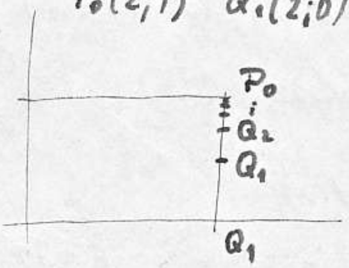
Pr.: $P_1(2;1)$ $\quad P_2(1;\frac{1}{2})$ $\quad P_3(\frac{3}{2};\frac{3}{4})$ $\quad P_4(\frac{5}{4};\frac{7}{8})$ $\quad \dots$



Post. bodov konverg. k P_2

$$P_n \to P_2$$

Pr.: $P_0(2;1)$ $\quad Q_1(2;0)$ $\quad Q_2(2;\frac{1}{2})$ $\quad Q_3(2;\frac{3}{4})$ $\quad \dots$



$$Q_n \to P_0$$

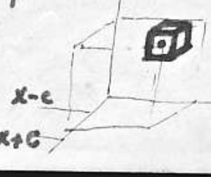
Pod symbolom E_3 budeme rozumieť trojrozmerný priestor. Množinu bodov $P(x;y;z)$.

Okolie bodu P_0 $O(P_0)$ – roz. mn. tých bodov z E_3, pre ktorých platí:

$$x_0 - c < x < x_0 + c$$
$$y_0 - c < y < y_0 + c \qquad \Big/ \ c \text{ ľubov. kladné číslo.}$$
$$z_0 - c < z < z_0 + c$$

(steny, hrany nepatr. do mn.!)

$|x - x_0| < c \quad |y - y_0| < c \quad |z - z_0| < c$



Pozn.: Pojmy otvor., uzavr., a ohran. množiny bodov definujeme podobne ako u E_2.

Majme prirodz. č. n. Pod symb. E_n rozumieme n rozmerný priestor. Je to súhrn vš. bodov ktoré majú n súradníc. $P(x_1, x_2, x_3, \dots x_n)$

Pod okolím bodu P_0 v n-rozm. priestoru rozumieme mn. vš. bodov z E_n, ktoré splnia podmienky:

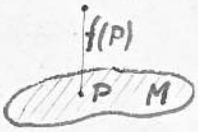
$$x_1^0 - c < x_1 < x_1^0 + c$$
$$x_2^0 - c < x_2 < x_2^0 + c$$
$$\dots$$
$$x_n^0 - c < x_n < x_n^0 + c$$

Pozn.: Pojmy otv., uzavr., množ. a konverg. Pojmy uzavr. bodov je def. podobne ako u E_2.

§2. Funkcie „n" premenných

Majme E_n a $M \neq 0$ ktorá je časťou priestoru $M \subset E_n$.

Def. 1.: Nech $P \subset M$. Ak každému P priradíme podľa urč. pravidla číslo $f(P)$, definovali sme tu f. n premenných na mn. M: $U = f(P)$



$P = (x_1; x_2 \dots x_n)$ .. $U = f(x_1; x_2 \dots x_n)$

M - je def. obor funkcie $f(P)$

Pr.: E_1 $u = f(x_1)$, $y = f(x)$

E_2 $u = f(x_1, x_2)$, $z = f(x; y)$; $z = 3x^2 + 2xy - 3$ $M = E_2$

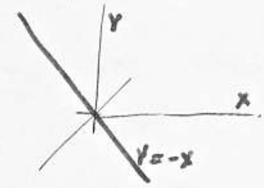
$f(3;1) = 3.3^2 + 2.3.1 - 3 = 30$

E_3 $u = f(x_1; x_2; x_3)$; $u = f(x; y; z)$ $= \frac{3xy}{2} - (x+z)^3 + \sqrt{5}$ $M = E_3$

$\frac{3xy}{z} - (x+z)^3 + \sqrt{5}$ $M = E_3$ okrem rov. (x,y)

Nech $M = E_2$.

$$z = \frac{2}{x+y} + 4 \quad ; \quad x+y \neq 0 \to y \neq -x$$



def. obor: celá rovina okrem bodov ktoré ležia na priamke $y = -x$.

Nech. $M \subset E_m$

D_2. Majme f. $f(P)$ s def. oborom M. F. $f(P)$ je spojitá v bode P_0 keď sú splnené tieto podmienky:

1.) $P_0 \in M$

2.) $\{P_m\} \subset M$

3.) ak z $P_m \to P_0$ vyplýva že $f(P_m) \to f(P_0)$

Podobne ako u f. 1 prem. dajú sa aj tu dokázať vety o spojitých funkciách viacpremenných.

D_3. Predpokl. že v každom $O(P_0)$ sa nachádzajú body patriace do M f. $f(P)$.
Ak z P_m konverg. k P_0 vyplýva že $f(P_m) \to L$, potom číslo L naz. limitou f. $f(P)$ v bode P_0.

$$\lim_{P \to P_0} f(P) = L$$

Pozn.: Namiesto symbolov P; P_0 môžeme písať aj príslušné súradnice bodov.
Ak f. $f(P)$ je spojitá v bode P_0, potom

$$\lim_{P_m \to P_0} f(P) = f(P_0)$$

Pr.: $\lim\limits_{\substack{x \to 1 \\ y \to 0 \\ z \to -5}} (3x^2y - z + 2) = 7$ $\quad \lim\limits_{P[x;y;z] \to P[1;0;-5]} (\ldots) =$

$\lim\limits_{[x;y;z] \to [1;0;-5]} (\ldots) =$

Pr.: $f(x;y) = \dfrac{x+y}{x}$; $x \neq 0$; M = rovina okrem osi y.

$\lim\limits_{\substack{x \to 0 \\ y \to 0}} \dfrac{x+y}{x}$

a.) Uvaž. konvergenciu po priamke $y = 0$, $\lim\limits_{\substack{x_m \to 0 \\ y \to 0}} \dfrac{x_m + y}{x_m} = \lim\limits_{x_m \to 0} \dfrac{x_m}{x_m} = 1$

b.) $y = x$; $\lim\limits_{\substack{x_m \to 0 \\ y_m \to 0}} \dfrac{x_m + y_m}{x_m} = \lim\limits_{x_m \to 0} \dfrac{2x_m}{x_m} = 2$ } f. nemá limitu $(1 \neq 2)$

Pr.: $f(x;y) = \ln(1-x^2-y^2)$

$1-x^2-y^2 > 0$

$-x^2-y^2 > -1$

$x^2+y^2 < 1$

M - otvorená

Funkcia 2 prem., spojitá na uzavr. a ohran. mn. má tieto vlastnosti: 1., je na tej mn. ohraničená tj. ex. také číslo K že pre vš. body $P_{(x;y)} \in M$ platí

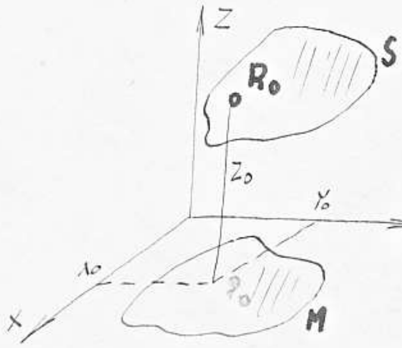
$$|f(x;y)| < K .$$

2. f. nadobúda na mn. M svojho maxima a minima.

28. IV. 1969.

§3 Graf funkcie 2. premenných.

Nech $M \subset E_2$; nech na M je def. f. $z = f(x;y)$.



$P_0(x_0 y_0) \subset M$ priradíme $z_0 = f(x_0;y_0)$

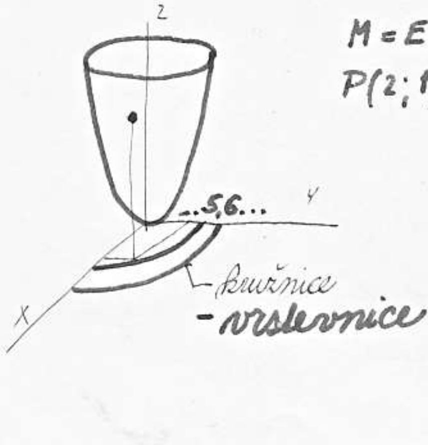
$R_0(x_0 y_0 z_0)$

Grafom f. $z = f(x;y)$ je plocha S, ktorú vytv. body R; pre ktoré platí:

1., $(x_0\, y_0) \in M$

2., $z_0 = f(x_0;y_0)$

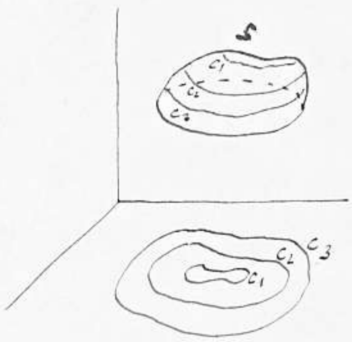
Pr.: $z = x^2+y^2$



$M = E_2$

$P(2;1)$ $z = 2^2+1^2 = 5$ $\Big\}$ $R(2,1,5)$

ak za z zvolíme urč. čísla (0; 1; 2 ... dostaneme rovnice vrstevníc



$z = f(x;y)$

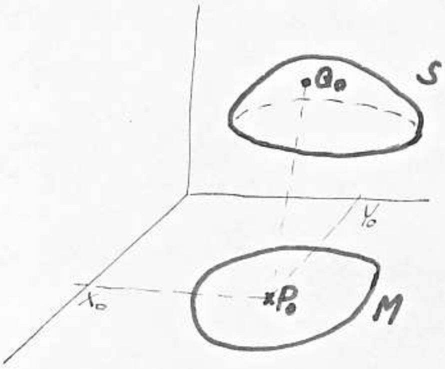
$c_1 = f(x_i;y_i)$

$c_2 = f(x_j;y)$

z vrst. sa dá vyčítať, ako rýchlo rastie graf.

4. § Parciálne derivácie

Máme $M \subset E_2$; na M def. $z = f(x;y)$.

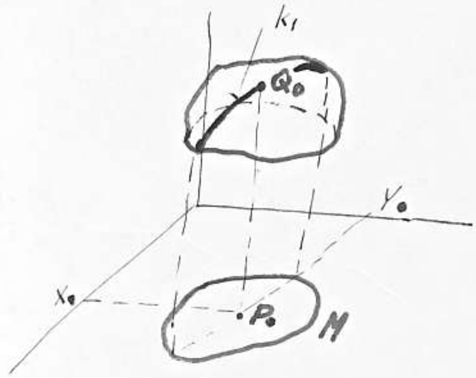


$z_0 = f(x_0\, y_0)$

$\underline{Q(x_0\, y_0\, z_0)}$

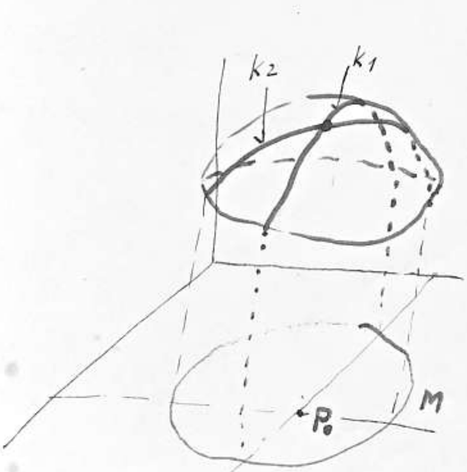
$z = f(x;y)$ položíme $y = y_0$ $\Big\}$ $z = f(x;y_0)$

Parciálna funkcia premennej x k funkcii $z = f(x;y)$



$y = y_0$ rovina rovnob. s (x,z)

Grafom parc. funkcie je krivka k_1 na ploche S. (leží v rov. $\parallel (zx)$

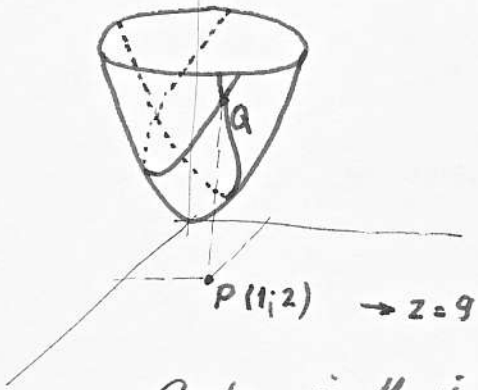


$z = f(x;y)$ $\quad x = x_0$ $\Big\}$ $z = f(x_0;y)$

Parc. funk. prem. y; f. $z = f(x;y)$

Graf. parc. funkcie krivky $\underline{k}$

Pr. $z = x^2 + 2y^2$



$\left.\begin{array}{l} z = x^2 + 2y \\ y = 2 \end{array}\right\} z = x^2 + 8$ – parabola na pl.

$\left.\begin{array}{l} z = x^2 + 2y \\ z = 1 \end{array}\right\} z = 1 + 2y^2$

Predp. že M je otvor. mn. a je časťou E_2.
Majme f.: $z = f(x;y)$ def. na M. a utvorme parc. funkciu $z = f(x;y_0)$; utvorme $\Delta_x z = f(x+\Delta x, y) - f(x;y)$

parciálny prírastok f. vzhľadom na premen. x.

$$\frac{\Delta_x z}{\Delta x} = \frac{f(x+\Delta x;y) - f(x;y)}{\Delta x}$$

D.1. Parciálna derivácia funkcie $z = f(x;y)$ v bode $P(x;y)$ je konečná limita (ak existuje):

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x;y) - f(x;y)}{\Delta x} = \frac{\partial z}{\partial x} \qquad \begin{bmatrix} z'_x & ; & z_x \\ f'_x & ; & f_x \end{bmatrix}$$

D.2. P. der. f. $z = f(x;y)$ podľa pr. y v bode $P(x;y)$ je kon. lim. (ak ex.):

$$\lim_{\Delta y \to 0} \frac{f(x;y+\Delta y) - f(x;y)}{\Delta y} = \frac{\partial z}{\partial y} \qquad \left[z'_y \ ; \ f'_y \right]$$

Pozn.: Výpočet p. der. podľa x (resp. y) prevádzam podľa pravidiel pre f. 1 premennej [dôkazy v z. sem

Pr.: $3x^2 + 2x \cdot y + e^y \cdot \ln x = z$

$$\frac{\partial z}{\partial x} = 6x + 2y + e^y \cdot \frac{1}{x}$$

$$\frac{\partial z}{\partial y} = 2x + e^y \ln x$$

$P(2;1)$

$$\frac{\partial z(2;1)}{\partial x} = 12 + 2 + e \cdot \frac{1}{2}$$

$$\frac{\partial z(2;1)}{\partial y} = 4 + e \ln 2$$

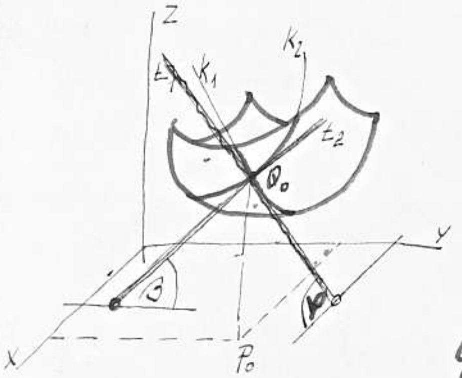
Pr: $z = y^x$ $\quad \frac{\partial z}{\partial x} = y^x \cdot \ln y$

$$\frac{\partial z}{\partial y} = x \cdot y^{x-1}$$

Geometrický význam parc. derivácie.

$M \subset E_2$ $\quad z = f(x;y)$

$P_0(x_0\ y_0) \in M \rightarrow z_0 f(x_0; y_0)$ $\Big\}$ $Q_0(x_0;y_0;z_0) \in S$



Pretože: $\dfrac{\partial z(x_0;y_0)}{\partial x} = \left[\dfrac{\partial z(x;y_0)}{\partial x}\right]_{x=x_0} \ldots$

je parc. der. z parc. f. $z=(x;y_0)$ podľa prem. x v bode x_0

Geom. význ: $\frac{\partial z}{\partial x}(x_0)$ je smernica dotyčnice ku krivke k_1. $\quad (\varphi = t_1 x+)$

$$\frac{\partial z(x_0;y_0)}{\partial y} = \left[\frac{\partial z(x_0;y)}{\partial y}\right]_{y=y_0} \qquad (\beta = t_2 ; y+)$$

Parc. der. funkcií viac premenných sa definujú analogicky ako parc. funkc. 2 premenných.

$u = f(x;y;z)$ $\quad \dfrac{\partial u}{\partial x} = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x;y;z) - f(x;y;z)}{\Delta x}$

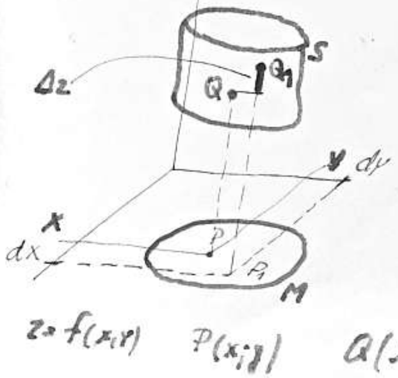
$$\frac{\partial u}{\partial z} = \lim_{\Delta z \to 0} \frac{f(x;y;z+\Delta z) - f(x;y;z)}{\Delta z}$$

§5. Differencovateľné funkcie viac premenných.

Uvaž. otv. mn. $M \subset E_2$ a na nej def. f. $z=f(x;y)$.

$\Delta z = f(x+\Delta x; y+\Delta y) - f(x;y)$ $\sim$ úplný prírastok funkcie 2. premenných.

[predp.: $P(x;y) \in M$; $P_1(x+\Delta x; y+\Delta y) \in M$]



$P_1(x+\Delta x; y+\Delta y)$ $\quad z=f(x+\Delta x; y+\Delta y)$ $\quad Q_1(x+\Delta x; y+\Delta y; z_1)$

$\Delta z = f(P_1) - f(P)$ $\begin{cases} >0 \\ <0 \\ =0 \end{cases}$

$z=f(x;y)$ $\quad P(x;y)$ $\quad Q(x;y;z)$

Ak f. $z=f(x;y)$ je spojitá v bode $P(x;y)$; potom lim:

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} f(x+\Delta x;\, y+\Delta y) = f(x;y) \quad \sim \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \Delta z = 0$$

a obrátene: ak $\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \Delta z = 0$ potom f. $z=f(x;y)$ je spojit $P(x;y)$

F. $z=f(x;y)$ je diferencovateľná v bode $P(x;y)$ ak jej úplný prírastok v $P(x;y)$ sa dá napísať vo tvare:

$$\Delta z = A(x;y)\Delta x + B(x;y)\Delta y + \alpha_1\Delta x + \alpha_2\cdot\Delta y; \quad (1)$$

ak α_1 a α_2 sú také že pri $\begin{matrix}\Delta x\to 0\\ \Delta y\to 0\end{matrix}$ aj $\begin{matrix}\alpha_1\to 0\\ \alpha_2\to 0\end{matrix}$

Výraz (1) sa zap. často aj vo tvare:

$$\Delta z = A(x;y)\Delta x + B(x;y)\Delta y + \alpha\cdot\rho \qquad \rho=\sqrt{(\Delta x)^2+(\Delta y)^2} \qquad \alpha=\frac{\alpha_1\Delta x+\alpha_2\Delta y}{\rho}$$

pričom ak $\left.\begin{matrix}\Delta x\to 0\\ \Delta y\to 0\end{matrix}\right\} \Rightarrow \rho\to 0 \Rightarrow \alpha\to 0$

Členy $A(x;y)$; $B(x;y)$ lineárne závisia od $\Delta x;\Delta y$ preto nazývame $A(x;y)\Delta x + B(x;y)\Delta y$ hlavnou časťou úpl. prírastku Δz.

$\alpha_1\Delta x+\alpha_2\Delta y$ ~ veličiny nekonečne malé (vyššieho rádu)

Pr: $z=x^2y$

$$\begin{aligned}\Delta z &= (x+\Delta x)^2(y+\Delta y)-(x^2y)\\ &=\left[x^2+2x\Delta x+(\Delta x)^2\right]\cdot(y+\Delta y) - x^2y =\\ &= x^2y + 2xy\Delta x + y(\Delta x)^2 + x^2\Delta y + 2x\Delta x\Delta y + \Delta y(\Delta x)^2 - x^2y\\ &= \underbrace{[2xy]\cdot\Delta x + [x^2]\Delta y}_{\text{hl. časť}} + \underbrace{y\cdot(\Delta x)^2 + 2x\Delta x\Delta y + (\Delta x)^2\Delta y}_{\text{velič. nekon. malé vyššieho rádu}}\end{aligned}$$

$$\left(\,[\underbrace{y\Delta x + 2x\Delta x}_{\alpha_1}]\Delta x + \underbrace{(\Delta x)^2}_{\alpha_2}\cdot\Delta y\right)$$

V1. Ak f. $f(x;y)$ je diferencovateľná v bode $P(x;y)$ potom je v bode P spoj.

Dôkaz: predp. že je dif.: $\Delta z = f(x+\Delta x; y+\Delta y) - f(x;y)$

ak je dif. $\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \Delta z = 0 \Rightarrow \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \{f(x+\Delta x; y+\Delta y) - f(x;y)\} = 0$

$z = \dots$ je spoj v $P(x;y)$ $\longleftarrow$ $\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} f(x+\Delta x; y+\Delta y) = f(x;y)$

V2. Ak $f(x;y)$ je dif. v bode $P(x;y)$ potom ex. parciálne derivácie $\frac{\partial f(P)}{\partial x}$ $\frac{\partial f(P)}{\partial y}$

Dôkaz: predp.: $f(x;y)$ je diferenc. — to znamená:

$$\Delta z = A(x;y)\Delta x + B(x;y)\Delta y + \alpha_1 \Delta x + \alpha_2 \Delta y$$

$$\left.\begin{matrix}\Delta x \to 0 \\ \Delta y \to 0\end{matrix}\right\} \Rightarrow \begin{matrix}\alpha_1 \to 0 \\ \alpha_2 \to 0\end{matrix}$$

položme: $\Delta y = 0 \dots \Delta_x z = A(x;y)\Delta x + \alpha_1 \cdot \Delta x$

$$\frac{\Delta_x z}{\Delta x} = A(x;y) + \alpha_1 \longrightarrow \lim_{\Delta x \to 0} \frac{\Delta_x z}{\Delta x} = \lim_{\Delta x \to 0} \{A(x;y) + \alpha_1\}$$

$$\frac{\partial z}{\partial x} = A(x;y)$$

$\Delta x = 0 \quad \dots \quad \frac{\partial z}{\partial y} = B(x;y)$

V1; V2 ~ sú nutné podmienky pre diferencovateľnosť f. $f(x;y)$

V3; (postačujúca podmienka) Ak v ľubov. okolí bodu $P(x;y)$ exist. parc. deriv. $\frac{\partial z}{\partial x}$; $\frac{\partial z}{\partial y}$ a sú spojité v bode $P(x;y)$ potom f. $f(x;y)$ je diferencovateľná v bode $P(x;y)$

Dôkaz: $\Delta z = f(x+\Delta x; y+\Delta y) - f(x;y) =$

$$= \underline{f(x+\Delta x; y+\Delta y) - f(x; y+\Delta y)} + \underline{f(x; y+\Delta y) - f(x;y)}$$

$f(x; y+\Delta y)$ ~ v int. $x \pm \Delta x$ splňa podm. Lagr. V.

$$\frac{f(x+\Delta x; y+\Delta y) - f(x; y+\Delta y)}{x+\Delta x - x} = f'_x(x+\Theta_1 \Delta x; y+\Delta y) \quad / \Theta \in \langle 0;1)$$

$$f(x+\Delta x; y+\Delta y) - f(x; y+\Delta y) = \Delta x \cdot f_x(x+\Theta_1 \Delta x; y+\Delta y)$$

$$f(x; y+\Delta y) - f(x;y) = \Delta y \cdot f'_y(x; y+\Theta_2 \Delta y) \quad / \text{Lagr. veta. } 0 < \Theta_2 < 1$$

$$\Delta z = f'_x(x+\Theta_1 \Delta x; y+\Delta y) \cdot \Delta x + f'_y(x; y+\Theta_2 \Delta y)\Delta y$$

$\longrightarrow$

podľa predpokladu: f_x' f_y' - sú spojité ; potom :

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} f_x'(x+\Theta_1 \Delta x; y+\Delta y) = f_x'(x;y)$$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} f_y'(x; y+\Theta_2 \Delta y) = f_y'(x;y)$$

$$f_x'(x+\Theta_1 \Delta x; y+\Delta y) = f_x'(x;y) + \alpha_1$$
$$f_y'(x; y+\Theta_2 \Delta y) = f_y'(x;y) + \alpha_2$$

/ ak $\left.\begin{matrix}\Delta x \to 0 \\ \Delta y \to 0\end{matrix}\right\} \Rightarrow \begin{matrix}\alpha_1 \to 0 \\ \alpha_2 \to 0\end{matrix}$

$$\Delta z = \{f_x'(x;y) + \alpha_1\} \cdot \Delta x + \{f_y'(x;y) + \alpha_2\} \cdot \Delta y =$$

$$= f_x'(x;y) \cdot \Delta x + f_y'(x;y) \Delta y + \alpha_1 \Delta x + \alpha_2 \Delta y \Rightarrow$$ f. je difer.-ná

F. diferencovateľná v každom bode oboru definície je diferencovateľná v celom obore.

§6. Dif. funkcie 2 premenných.

Nech $z = f(x;y)$ je diferencovateľná v bode $P(x;y) \Rightarrow$

$$\Delta z = \frac{\partial f(P)}{\partial x} \cdot \Delta x + \frac{\partial f(P)}{\partial y} \Delta y + \alpha_1 \Delta x + \alpha_2 \Delta y$$

v ľubov. bode:

$$\Delta z = \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y + \alpha_1 \Delta x + \alpha_2 \Delta y$$

$$dz = \frac{\partial z}{\partial x} \cdot \Delta x + \frac{\partial z}{\partial y} \Delta y$$

~ úplný dif. f. $f(x;y)$ (totálny diferenciál)

Utvorme: $\Delta z - dz = \alpha_1 \Delta x + \alpha_2 \Delta y$

$$\left.\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} (\Delta z - dz) = 0 \right\} \quad \Delta z \doteq dz$$

$$f(x+\Delta x; y+\Delta y) - f(x;y) \doteq \frac{\partial z}{\partial x} \Delta x + \frac{\partial z}{\partial y} \cdot \Delta y$$

$$f(x+\Delta x; y+\Delta y) \doteq f(x;y) + dy$$

Pr.: 1. $z = e^{xy}\cdot \ln x + \cos xy$

$$\frac{\partial z}{\partial x} = y\cdot e^{xy}\cdot \ln x + \frac{e^{xy}}{x} - y \sin xy$$

$$\frac{\partial z}{\partial y} = x e^{xy}\cdot \ln x - x \sin xy \qquad dz = \left(y \ln x\, e^{xy} + e^{xy}/x - y \sin xy\right)\Delta x + \left(x \ln x\cdot e^{xy} - x \sin xy\right)\Delta y$$

Pr.: 2. vyp: $0{,}97^{1{,}03}$

$$z = y^x \doteq 1^1 + \dots$$

$$A[1;1] \rightarrow B[1{,}03;\, 0{,}97] \qquad \Delta x = 1{,}03 - 1 = 0{,}03$$
$$\Delta y = 0{,}97 - 1 = -0{,}03$$

$$dz = y^x \ln y\, \Delta x + x y^{x-1} \Delta y = 1^1 \ln 1 \cdot 0{,}03 + 1\cdot 1^{1-1}(-0{,}03) \doteq$$
$$0{,}97^{1{,}03} \doteq 1 - 0{,}03 \doteq 0{,}97.$$

§7 Derivácie a diferenciál zložených f.

Nech f. $z = f(x;y)$ je taká, že $x;y$ sú zase funkcie;

1. Nech $z = f(x;y)$; $x = g_1(t)$ $y = g_2(t)$

Nech $g_1(t)$ a $g_2(t)$ sú difer. v bode (t)

Nech f. $f(x;y)$ je dif. v bode $P(x;y)$; potom aj f. $z = f[g_1(t);\, g_2(t)]$ je dif. v bode $\underline{t}$ a platí:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} \qquad (1)$$

Dôkaz: Podľa predp. vety: $\frac{dx}{dt} = \lim\limits_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$ $\qquad \frac{dy}{dt} = \lim\limits_{\Delta t \to 0}\frac{\Delta y}{\Delta t}$

— 〃 — $\Delta z = f'_x(x;y)\Delta x + f'_y(x;y)\Delta y + \alpha_1 \Delta x + \alpha_2 \Delta y$

$$\frac{\Delta z}{\Delta t} = f'_x(x;y)\frac{\Delta x}{\Delta t} + f'_y(x;y)\frac{\Delta y}{\Delta t} + \alpha_1\frac{\Delta x}{\Delta t} + \alpha_2\frac{\Delta y}{\Delta t}$$

$$\lim_{\Delta t\to 0}\frac{\Delta z}{\Delta t} = \frac{\partial z}{\partial x}\cdot\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t} + \lim_{\Delta t\to 0}\frac{\Delta y}{\Delta t}\cdot\frac{\partial z}{\partial y} + \alpha_1\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t} + \alpha_2\lim_{\Delta t\to 0}\frac{\Delta y}{\Delta t}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} + 0\cdot\frac{dx}{dt} + 0\,\frac{dy}{dt}$$

Pr.: $z = x^2 + 2y$; $x = 3t+1$ $\quad y = t^2 + \sin t$

$z = (3t+1)^2 + 2t^2 + 2\sin t$

$$\frac{dz}{dt} = 2x\cdot 3 + 2(2t + \cos t)$$
$$\frac{dz}{dt} = 6(3t+1) + 4t + 2\cos t = 22t + 2\cos t + 6$$

Pr.: $z = e^{t^2+\sin t} + \ln(t^2+\sin t) - \frac{1}{\sqrt{\operatorname{tg} t}} + \ln \operatorname{tg} t$ $\quad /\quad x = t^2 + \sin t \quad y = \operatorname{tg} t$

$$z = e^x + \ln x - \frac{1}{\sqrt{y}} + \ln y$$

$$\frac{dz}{dt} = \left(e^x + \frac{1}{x}\right)(2t + \cos t) + \frac{1}{y} + \frac{1}{2}\cdot\frac{1}{\sqrt{y^3}}\cdot\frac{1}{\cos^2 t}$$

$$\frac{dz}{dt} = \left(e^{t^2+\sin t} + \frac{1}{t^2+\sin t}\right)(2t+\cos t) + \frac{1}{\operatorname{tg} t} + \frac{1}{2\sqrt{\operatorname{tg}^3 t}}\cdot\frac{1}{\cos^2 t} \;;$$

2.) $z = f(x;y) \qquad y = g(x)$

predp. $f(x;y)$ je diferenc. v $P(x;y)$ | $g(x)$ je dif. v bode x

$z = f[x; g(x)] = F(x)$

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx} \qquad (2)$$

3.) $z = f(x;y) \qquad x = g_1(u;v) \qquad y = g_2(u;v)$

$z = f[g_1(u;v); g_2(u;v)] = H(u;v)$

$$\left.\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} \end{aligned}\right\} \qquad (3)$$

Pr.: $z = x + e^y \qquad x = u + 3v^2 \qquad y = \ln u + 5v$

$$\frac{\partial z}{\partial u} = 1 \cdot 1 + e^y \cdot \frac{1}{u} = 1 + \frac{e^{\ln u + 5v}}{u}$$

$$\frac{\partial z}{\partial v} = 1 \cdot 6 \cdot v + e^y \cdot 5 = 6v + 5e^{\ln u + 5v}$$

$z = f(x;y) \qquad dz = \frac{\partial z}{\partial x} \cdot \Delta x + \frac{\partial z}{\partial y} \cdot \Delta y$ ~ pre jednoduchú funkciu

zoberme: $z = x \qquad \left.\begin{aligned} dz &= 1 \cdot \Delta x \\ dz &= dx \end{aligned}\right\} \Delta x = dx$

zvolme: $z = y \ldots \qquad \ldots \Delta y = dy$

Ak f. z nie je zložitá; potom: $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} \cdot dy$

$z = f(x;y) \qquad \begin{aligned} x &= g_1(u;v) \\ y &= g_2(u;v) \end{aligned} \qquad dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \qquad z = f[g_1(u;v); g_2(u;v)] = F(u;v)$

$dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} \cdot dv$ ← dosad. z rovn (3)

$$dz = \left(\frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u}\right) du + \left(\frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v}\right) \cdot dv =$$

$$= \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial u} \cdot du + \frac{\partial x}{\partial v} dv\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv\right) = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy$$

To že tvar dif. je rovnaký ako pre jednod.) tak aj pre zložené f. (forma dif. nezávisí od toho či $x;y$ sú premenné, alebo funkcie) nazývame invariantnosťou formy úplného diferenciálu.

$\left[\begin{aligned} dx &= \Delta x \\ dy &= \Delta y \end{aligned}\right.$ ~ len keď f. je jednoduchá $]$

§8 Parciálne der. a diferenciály vyšších rádov

ak f $z = f(x;y)$ má parc. der. 1r. v $P(x;y)$ potom môžeme utvoriť parc. der. 2.r ....

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \qquad z''_{xx} \qquad f''_{xx} \qquad \frac{\partial^{m+n} z}{\partial z^{m+n}} = \frac{\partial^{m+n} z}{\partial x^m \partial y^n} \text{ – r. der.}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x;\partial y} \qquad z''_{xy} \qquad f''_{xy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y\,\partial x} \qquad z''_{yx} \qquad \vdots$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} \qquad z''_{yy} \qquad \vdots$$

Pr: $z = 3x^2y + \cos x + e^y$

$\frac{\partial z}{\partial x} = 6xy - \sin x$ $\qquad \frac{\partial^2 z}{\partial x^2} = 6y - \cos x \qquad \frac{\partial^2 z}{\partial x \partial y} = 6x$

$\frac{\partial z}{\partial y} = 3x^2 + e^y$ $\qquad \frac{\partial^2 z}{\partial y^2} = e^y \qquad \frac{\partial^2 z}{\partial y \partial x} = 6x$

Dá sa dokázať že ak f. $z = f(x;y)$ má v $O_{P(x;y)}$ parc. derivácie zmiešané ; 2r ; a tieto sú spojité v bode P ; potom platí:

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

$z = f(x;y)$ – je diferencovateľná v $P(x;y)$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}\cdot dy$$

$$d^2z = d(dz) = d\left(\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\right) = d\left(\frac{\partial z}{\partial x}dx\right) + d\left(\frac{\partial z}{\partial y}dy\right) =$$

$$= \left(\frac{\partial^2 z}{\partial x^2}dx + \frac{\partial^2 z}{\partial y \partial x}dy\right)dx + \left(\frac{\partial^2 z}{\partial x \partial y}dx + \frac{\partial^2 z}{\partial y^2}dy\right)dy =$$

$$= \frac{\partial^2 z}{\partial x^2}dx^2 + 2\frac{\partial^2 z}{\partial x \partial y}\cdot dx\,dy + \frac{\partial^2 z}{\partial y^2}dy^2 ;$$

$$d^2 z = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^2 z \quad \text{~ symbolický zápis.}$$

$$d^n z = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^n z \quad \text{~ pre } n \text{ prirodzené } (1;2;3\ldots)$$

Pr: $n = 3$ $\qquad d^3 z = \frac{\partial^3 z}{\partial x^3}dx^3 + 3\frac{\partial^3 z}{\partial x^2 \partial y}dx^2dy + 3\frac{\partial^3 z}{\partial x \partial y^2}dx\,dy^2 + \frac{\partial^3 z}{\partial y^3}dy^3$

9.§ Funkcie v implicitnom tvare

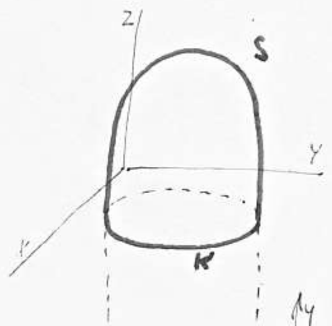
Majme f. $y=f(x)$, def. obor M.

$y=f(x)$ – explicitný tvar $\left[\begin{matrix} x = \dots \\ y = \dots \end{matrix} \right]$

[Kružnica, elipsa… nie je f.]

Ak $y=f(x)$ je f., potom jej graf rovnobežky s y pretína najviac v 1 bode.

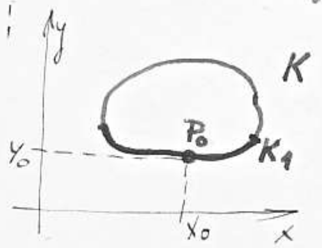
Majme f. $z=F(x;y)$ (1)



$\underbrace{z=0 \sim \text{rovnica rov}(x;y) \qquad z=F(x;y)}$

(2) $0=F(x;y)$ ~ rovnica krivky K

Ak plocha S nereže rovinu $(x\,y)$ potom kr. K neexistuje



Zvolme $P_0(x_0;y_0)\in k$;

potom $F(P_0)=0$

Vyberme kr. $K_1 \subset K$; tak aby:

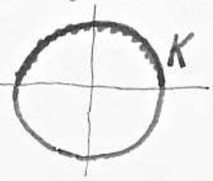
1, $P_0 \in K_1$

2, K_1 bola grafom f. premennej x

(K_1 sa môže s y a s jej rovnobežkou pretnúť najviac v 1 bode.)

Ak K_1 je grafom f. $y=f(x)$; potom hovoríme, že f. $y=f(x)$ je daná v implicitnom tvare rovnicou $F(x;y)=0$

Pr: $x^2+y^2-4=0$



rovnica

$y_{1,2}=\pm\sqrt{4-x^2}$ funkcia

$y_1=\sqrt{4-x^2} = f_1(x)$

$y_2=-\sqrt{4-x^2} = f_2(x)$

V1. Nech f. $z = F(x;y)$ je definovaná v $O_{P_0}(x_0;y_0)$; a nech platí:
1, $F(P_0) = 0$
2, $F(x;y)$ má v P_0 spojité parc. derivácie 1 r.
3, $\dfrac{\partial F(P_0)}{\partial y} \neq 0$

potom ex. v urč. okolí x_0 jediná spojitá f. $y = f(x)$; ktorá je daná v implicitnom tvare rovnicou $F(x;y) = 0$ a pre ktorú platí: ~~F(x;y)~~ $y_0 = f(x_0)$.

V2. Za predp. 1 – 3 V1. platí: f. $y = f(x)$ má der. v bode x_0.

$$f'(x_0) = -\frac{\frac{\partial F(x_0 y_0)}{\partial x}}{\frac{\partial F(x_0 y_0)}{\partial y}} \qquad y'(x_0) = -\frac{F'_x(P_0)}{F'_y(P_0)}$$

v ľubov. bode: $y' = -\dfrac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$;

Dôkaz.

grafom $F(x;y) = 0$ je kr. K
-"- $y = f(x)$ -"- K_1

$\underbrace{\Delta z = F(P) - F(P_0)}_{F(P)=0}$ / však podľ. predp. je $F(P_0) = 0$; uvaž. $F(x;y) = 0$ [sme v rov.]

prírastok = 0

$$\sim \Delta z = dz + \alpha_1 \Delta x + \alpha_2 \Delta y = 0 \quad \left| \begin{matrix} \Delta x \to 0 \\ \Delta y \to 0 \end{matrix} \right\} \Rightarrow \begin{matrix} \alpha_1 \to 0 \\ \alpha_2 \to 0 \end{matrix}$$

$$0 = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy + \alpha_1 \Delta x + \alpha_2 \Delta y$$

v bode P_0 :

$$F'_x(P_0)\Delta x + F'_y(P_0)\Delta y + \alpha_1 \Delta x + \alpha_2 \Delta y = 0 \quad / \cdot \frac{1}{\Delta x}$$

$$0 = F'_x(P_0) + \frac{F'_y(P_0)\Delta y}{\Delta x} + \alpha_1 + \alpha_2 \frac{\Delta y}{\Delta x}$$

$$0 = F'_x(P_0) + F'_y(P_0) \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} + \lim_{\Delta x \to 0} \alpha_1 + \lim_{\Delta x \to 0} \alpha_2 \cdot \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

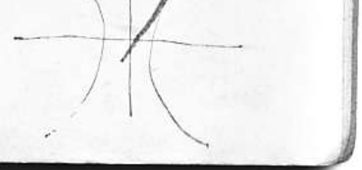
$$0 = F'_x(P_0) + F'_y(P_0) \cdot f'(x_0) + 0 + 0 \cdot f'(x_0)$$

$$f'(x_0) = -\frac{F'_x(P_0)}{F'_y(P_0)}$$

Pr. Je daná k. $x^2 - y^2 = 1$
Nap. rovn. dot. v bode $x_0 = 1$

$P(1;0)$ $\quad f'(x) = \dfrac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\dfrac{2x}{-2y} = \dfrac{x}{y}$ $\quad k = \dfrac{1}{0}$

V bode $P(2;\sqrt{3})$ $\quad k = \dfrac{2}{\sqrt{3}}$ $\quad \Big\}\ y - \sqrt{3} = \dfrac{2}{\sqrt{3}}(x - 2)$



$F(x;y;z) = 0$ môže obsah. implic. $z = f(x;y)$

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = \frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

§10. Vektorová f. a jej derivácie

Majme $f(x)$ def. na M. $y = f(x)$

$\bar{f}(x)$ K

Každému bodu množ. priraď. vektor : def. sme vekt. funkciu na skalárn. premennej.

$$\bar{f}(x) = f_1(x)i + f_2(x)j + f_3(x)k$$

$$\bar{g}(x;y) = g_1(x;y)i + g_2(x;y)j + g_3(x;y)k$$

$$\bar{f}'(x) = f_1'(x)i + f_2'(x)i + f_3'(x)k$$

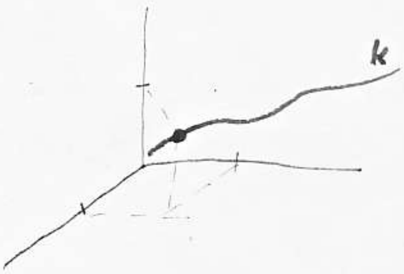
$$\frac{\partial \bar{g}(x;y)}{\partial x} = \frac{\partial g_1(x;y)}{\partial x}i + \frac{\partial g_2(x;y)}{\partial x}j + \frac{\partial g_3(x;y)}{\partial x}k$$

$$\bar{f}(t) = f_1(t)i + f_2(t)j + f_3(t)k \qquad (1)$$ t – parameter

$$\left.\begin{aligned} x &= f_1(t) \\ y &= f_2(t) \\ z &= f_3(t) \end{aligned}\right. \qquad (2)$$

Sústava 2 v priestore nám určuje priestorovú krivku za predp. ak $f_1(t) \cdots f_3(t)$ sú spoj. f. premennej t; a aspoň 1 z prvých derivácií podľa t $\neq 0$

Sústava 2 sa dá vyj. aj vektorovou rovnicou (1)



Ak vš. derivácie $= 0$ ~ sú konštanty a to je len bod.

Δ - operátor nabla

$$\Delta = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

$u = f(x;y;z)$ $\quad \Delta f = \left(i\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) f(x;y;z)$

$$\Delta f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} \sim \text{je to vektorová f.}$$

Δf = gradient $f(x;y;z)$ [grad.]

2.) Pouš. na vektorovú f. :

$$\bar{f}(x;y;z) = if_1(x;y;z) + jf_2(x;y;z) + kf_3(x;y;z)$$

skal. súčin:

$$\Delta\bar{f} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot(if_1 + jf_2 + kf_3) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\Delta\bar{f} = \text{div}\,\bar{f} \quad (\text{divergencia funkcie})$$

Divergencia z vektorovej funkcie je skalárna.

3.) Utvorme vektorový súčin:

$$\Delta\times\bar{f} = \left(i\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\times(if_1 + jf_2 + kf_3)$$

$$\Delta\times\bar{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = i\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - j\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + k\left(-\frac{\partial f_1}{\partial y} + \frac{\partial f_2}{\partial x}\right)$$

$$\Delta\times\bar{f} = \text{rot.}\,\bar{f} \quad (\text{rotácia f.})$$

Rotácia vektorovej funkcie je opäť vektorová funkcia.

4.) $\nabla = \Delta.\Delta$ ~ Laplaceov operátor

$$\nabla = \left(i\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Pr: $f(x;y;z) = 3x^2y + z\cos x + z^y$

$\underline{\text{grad}\, f(x;y;z)} = i(6xy - z\sin x) + j(3x^2 + zy\ln z) + k.\cos x$

$f(x;y;z) = i\,3xyz - je^{xy}\sin z + k\,2x$

$\text{div.}\,\bar{f}(x;y;z) = 3yz + (-xe^{xy}\sin z) + 0 = 3yz - xe^{xy}\sin z$

$$\text{rot.}\,\bar{f}(x;y;z) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xyz & -e^{xy}\sin z & 2x \end{vmatrix} = i(0 + e^{xy}\cos z) - j(2 - 3xy) + k(-ye^{xy}\sin z - 3xz) =$$

$$= ie^{xy}\cos z + j(3xy - 2) - k(3xz + ye^{xy}\sin z).$$

§ 11. Dotyčnica priestorovej krivky a dotyková rovina plochy.

Majme vekt. f.: $\bar{r}(t) = i\,x(t) + j\,y(t) + k\,z(t)$ (1)

ak $t = t_0$; tak $\bar{r}(t_0)$ - je vektor

$\bar{r}(t_0) = i\,x(t_0) + j\,y(t_0) + k\,z(t_0)$

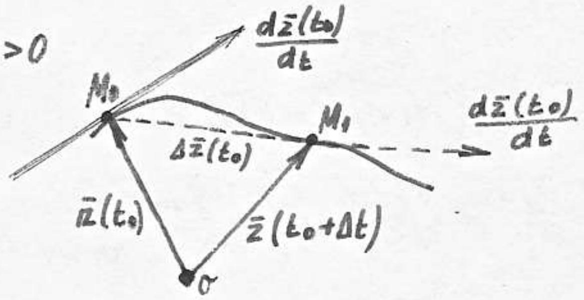
$\Delta \bar{r}(t) = \bar{r}(t_0 + \Delta t) - \bar{r}(t_0)$

$$\frac{d\bar{r}(t_0)}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \bar{r}(t_0)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\bar{r}(t_0+\Delta t) - \bar{r}(t_0)}{\Delta t}$$

pre ľubov. t : $$\frac{d\bar{r}(t)}{dt} = \lim_{\Delta t \to 0} \frac{\bar{r}(t+\Delta t) - \bar{r}(t)}{\Delta t}$$

$$\frac{d\bar{r}(t_0)}{dt} = i\,\frac{dx(t_0)}{dt} + j\,\frac{dy(t_0)}{dt} + k\,\frac{dz(t_0)}{dt}$$

1.) $\Delta t > 0$



$\bar{r}(t_0) + \Delta \bar{r}(t_0) = \bar{r}(t_0 + \Delta t)$

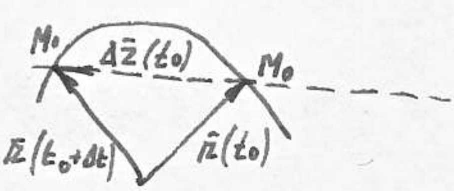
$\Delta \bar{r}(t_0) = \bar{r}(t_0 + \Delta t) - \bar{r}(t_0)$

$\frac{\Delta \bar{r}(t_0)}{\Delta t}$ ... v. má smer vektora $\Delta \bar{r}(t_0)$

$$\frac{d\bar{r}(t_0)}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \bar{r}(t_0)}{\Delta t}$$

$\frac{d\bar{r}(t_0)}{dt}$ ... má smer dotyčnice v b. M_0

2.) $\Delta t < 0$



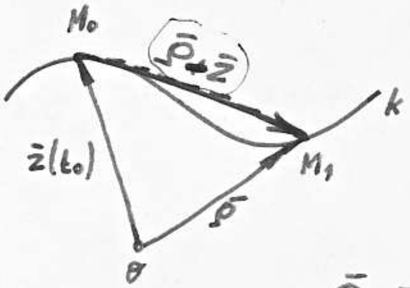
$\frac{\Delta \bar{r}(t_0)}{\Delta t}$ .... opačný smer $\Delta \bar{r}(t_0)$

$\frac{d\bar{r}(t_0)}{dt}$ ... opäť má smer dotyčn. v bode M_0

a.) Dotyčnica k priestorovej krivke

$$x = x(t) \quad y = y(t) \quad z = z(t) \qquad (1)$$

Predpokl., že f. $x(t)$ .... sú spojité a majú aspoň 1. deriváciu $1^r \neq 0$



$M_0(x_0; y_0; z_0)$ - pevný
$M_1(x; y; z)$ - ľub. premenný

$\bar{r} = \bar{r}(x_0; y_0; z_0) \quad \bar{\rho} = \bar{\rho}(x; y; z)$

$$\bar{\rho} - \bar{r} = s\frac{d\bar{r}}{dt} \qquad (1) \quad \text{vektorová rovnica dotyčnic.}$$

$$\bar{\rho} = \bar{r} + s\frac{d\bar{r}}{dt}$$

$$\left.\begin{aligned} x &= x(t_0) + s\frac{dx(t_0)}{dt} \\ y &= y(t_0) + s\frac{dy(t_0)}{dt} \\ z &= z(t_0) + s\frac{dz(t_0)}{dt} \end{aligned}\right| \quad \text{stručne:} \quad \begin{aligned} x &= x_0 + s\frac{dx}{dt} \\ y &= y_0 + s\frac{dy}{dt} \\ z &= z_0 + s\frac{dz}{dt} \end{aligned}$$

$$\frac{x-x_0}{\frac{dx}{dt}} = \frac{y-y_0}{\frac{dy}{dt}} = \frac{z-z_0}{\frac{dz}{dt}}$$

$$\frac{x-x^0}{x'_0} = \frac{y-y^0}{y'_0} = \frac{z-z^0}{z'_0}$$

Pr.: Nap. rovnicu dotyčnic ku krivke: $x = t - \sin t$, $y = 1 - \cos t$, $z = 4\sin\frac{t}{2}$ v bode $t_0 = \frac{\pi}{2}$

$$\left.\begin{aligned} x_0 &= \frac{\pi}{2} - \sin\frac{\pi}{2} = \frac{\pi}{2} - 1 \\ y_0 &= 1 - \cos\frac{\pi}{2} = 1 \\ z_0 &= 4\sin\frac{\pi}{4} = 2\sqrt{2} \end{aligned}\right\} \quad M_0\left(\frac{\pi}{2}-1;\ 1;\ 2\sqrt{2}\right)$$

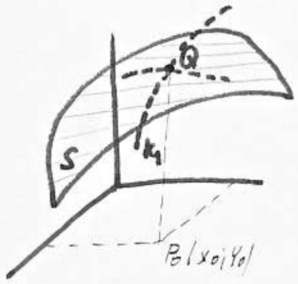
$$\begin{aligned} x' &= 1 - \cos t \\ y' &= \sin t \\ z' &= 2\cos\frac{t}{2} \end{aligned} \qquad \begin{aligned} x'_0 &= 1 - \cos\frac{\pi}{2} = 1 \\ y'_0 &= \sin\frac{\pi}{2} = 1 \\ z'_0 &= 2\cos\frac{\pi}{4} = \sqrt{2} \end{aligned}$$

rovnica dotyčnic: $$\frac{x - \frac{\pi}{2} + 1}{1} = \frac{y-1}{1} = \frac{z - 2\sqrt{2}}{\sqrt{2}}$$

b. Dotyková rovina k ploche:

Predpokl: $f(x;y)$ $\quad$ $P_0(x_0;y_0)$ spojité parc. derivácie

Ozn. $\underline{S}$ plochu; ktorá je grafom f. $f(x;y)$



$Q_0(x_0;y_0;z_0)$

$z=f(x;y;z)$

$y=y_0$ ...... parc. f. prem x ... jej graf je $k_1 \parallel (x;z)$

$x=x_0$ ..... — " — y .... — " — $k_2 \parallel (y;z)$

Postup: Nájd. vektor $\bar{n}_1$; majúci smer dot. ku krivke k_1 v bode Q.

— " — $\bar{n}_2$ — " — k_2 — " —

Tieto dva vektory a bod Q jednoznačne urč. dot. rovinu.

$\bar{n} = \bar{n}_1 \times \bar{n}_2$

zoberieme f. $z_1 = f(x;y_0)$ .... k_1 vyjadr. v par. súradniciach:

$x=t$ $\qquad$ $x'=1$

$y=y_0$ $\qquad$ $y'=0$

$z=f(t;y_0)$ $\qquad$ $z'=\frac{\partial f}{\partial x}\cdot \underbrace{\frac{dx}{dt}}_{1} = f'_x$ $\qquad$ $\bar{n}_1=(1;0;f'_x(t_1))$

$y=t$ $\qquad$ $y'=1$

$x=x_0$ $\qquad$ $x'=0$

$z=f(x_0;t)$ $\qquad$ $z'=f'_y$ $\qquad$ $\bar{n}_2=(0;1;f'_y(t_2))$

$$\bar{n}=\bar{n}_1\times\bar{n}_2=\begin{vmatrix} i & j & k \\ 1 & 0 & f'_x(t) \\ 0 & 1 & f'_y(t)\end{vmatrix} = i\{-f'_x(t_0)\} - j\{f'_y(t_0)\} + k(1) = -i f'_x(t_0) - j f'_y(t_0) + k$$

$Q_0(x_0;y_0;z_0)$ ~ dotykový bod

$$-f'_x(t_0)\cdot(x-x_0) - f'_y(t_0)\cdot(y-y_0) + 1(z-z_0) = 0 \quad /\cdot(-1)$$

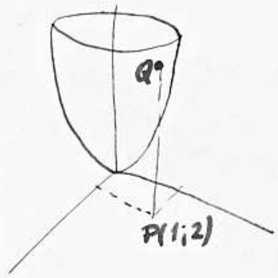
$$\frac{\partial f}{\partial x}(x-x_0) + \frac{\partial f}{\partial y}(y-y_0) - (z-z_0) = 0$$

$$\frac{\partial f(P_0)}{\partial x}\cdot(x-x_0) + \frac{\partial f(P_0)}{\partial y}(y-y_0) - (z-z_0) = 0$$

Ak je plocha daná implicitne rovnicou $F(x;y;z)=0$ potom sa dá dokázať, že rovnica dotykovej roviny má tvar:

$$\frac{\partial F(Q_0)}{\partial x}(x-x_0) + \frac{\partial F(Q_0)}{\partial y}(y-y_0) + \frac{\partial F(Q_0)}{\partial z}(z-z_0) = 0$$

Pr.: Vyp. rovnicu dotyk. roviny paraboloidu $z = 2x^2+4y^2$ v bode $Q(1;2;18)$



$$\frac{\partial z}{\partial x} = 4x \qquad \frac{\partial f(P_0)}{\partial x} = 4$$

$$\frac{\partial z}{\partial y} = 8y \qquad \frac{\partial f(P_0)}{\partial y} = 16$$

$$4(x-1) + 16(y-2) + -(z-18) = 0$$

$$4x + 16y - z - 18 = 0 \quad \sim \text{rovn. plochy}$$

Nap. rovn. dotyk. roviny ku elipsoidu $4x^2+9y^2+36z^2 = 36$ $Q\left(2;1;\frac{\sqrt{11}}{6}\right)$

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

$$\frac{\partial F}{\partial x} = \dots = 8x \quad \Big| \quad 16$$

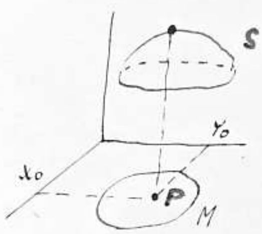
$$\frac{\partial F}{\partial y} = \dots = 18y \quad \Big| \quad 18 \qquad 16(x-2)+18(y-1)+12\sqrt{11}\left(z-\frac{\sqrt{11}}{6}\right) = 0$$

$$\frac{\partial F}{\partial z} = \dots = 12z \quad \Big| \quad 12\cdot\sqrt{11}:6 \qquad 8(x-2)+9(y-1)+6\sqrt{11}\left(z-\frac{\sqrt{11}}{6}\right) = 0.$$

§12. Lokálne extr. f. 2 premenných

12.V. 1969.

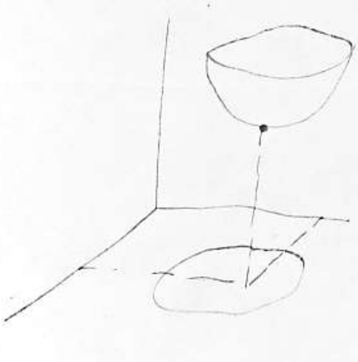
Nech $M \subset E_2$ a nech $z = f(x;y)$ má spoj. derivácie 1 a 2^r v $P_0(x_0;y_0) \in M$. Hovoríme že f. $f(x;y)$ má v bode P_0 lok. maximum ak hodnoty tejto f. v každom bode z $O_{P_0} \neq P_0$ sú menšie než v bode P_0.



$$f(x_0+\Delta x; y_0+\Delta x) < f(x_0;y_0)$$

$$f(x_0+\Delta x; y_0+\Delta y) - f(x_0;y_0) < 0 \qquad (1)$$

Hov. že $f(x;y)$ má lok. min v P_0; ak. hodnoty tejto f. v každom bode okolia P_0 ~~≠~~ P_0 sú väčšie než v samotn. bode P_0.



$$f(x_0+\Delta x; y_0+\Delta y) > f(x_0;y_0)$$

$$f(x_0+\Delta x; y_0+\Delta y) - f(x_0;y_0) > 0 \qquad (2)$$

Lok. max. a lok. minimum – lok. extrémy.

V1. Nutná podmienka pre extrém

Nech $f(x;y)$ má v bode $P_0(x_0;y_0)$ parc. der. 1ᵛ potom nutn. podm. aby f. $(x;y)$ mala v P_0 extrém je: $\frac{\partial f(P_0)}{\partial x}=0$; $\frac{\partial f(P_0)}{\partial y}=0$

Dôkaz: Predp. $f(x;y)$ má v $P_0(x_0;y_0)$ l. extrém. nech je to l. Max.

$$\left.\begin{array}{l} f(x_0+\Delta x;\ y_0+\Delta y)-f(x_0;y_0)<0 \\ f(x_0+\Delta x;\ y_0)-f(x_0;y_0)<0 \end{array}\right\} \overset{\text{parc. prír.}}{\Delta z_x} = f(x_0+\Delta x;\ y_0)-f(x_0;y_0)$$

Z 1. sem.: nutn. podm. pre extr. aby der. parc. funkc. $=0$

$$z'_x(x_0)=0 \rightarrow \frac{\partial f(P_0)}{\partial x}=0$$

Analogicky sa dá dok.: ak $f(x_0;y_0+\Delta y)-f(x_0;y_0)<0$

$$z'_y(y_0)=0 \rightarrow \frac{\partial f(P_0)}{\partial y}=0$$

Pozn.: Body v ktorých $\frac{\partial f}{\partial y}$; $\frac{\partial f}{\partial y}=0$ nazývame stacionárnymi bodmi. f. môže ale nemusí mať extrém.

V2. Postačujúca podmienka

Nech $f(x;y)$ má v $P_0(x_0;y_0)$ spoj. parc. der. 1ᵛ a 2ᵛ.
Nech $\frac{\partial f(P_0)}{\partial x}=0$ $\frac{\partial f(P_0)}{\partial y}=0$

Označme: $\frac{\partial^2 f(P_0)}{\partial x^2}=A$ $\frac{\partial^2 f(P_0)}{\partial x\partial y}=B$ $\frac{\partial^2 f(P_0)}{\partial^2 y}=C$

Potom 1.) $AC-B^2>0$ – l. extrém ~~existuje~~

2.) $AC-B^2=0$ – nerozhodný prípad

Dôkaz (nerobíme)

Nech je vz. dok.: $AC-B^2>0 \sim \frac{\partial^2 f(P_0)}{\partial x^2}\cdot\frac{\partial^2 f(P_0)}{\partial y^2}-\left(\frac{\partial^2 f(P_0)}{\partial x\partial y}\right)^2>0$

$AC-B^2>0 \Rightarrow AC>B^2 \Rightarrow AC>0 \Rightarrow$ A a C majú rovnaké znam.

Pr.: Nájd. lok. extr. f.: $z=2x^3+xy^2+5x^2+y^2$

$$\frac{\partial z}{\partial x}=6x^2+y^2+10x \qquad 6x^2+10x+y^2=0$$
$$\frac{\partial z}{\partial y}=2xy+2y \qquad y(x+1)=0$$

1.) $y=0 \rightarrow x_1=0,\ x_2=-\frac{5}{3}$ } $P(0;0)$ $R(-\frac{5}{3};0)$

2.) $(x+1)=0$, $x=-1 \rightarrow y_1=-2,\ y_2=+2$ } $S(-1;2)$ $D(-1;-2)$

$\frac{\partial^2 z}{\partial x^2} = 12x+10$ $\quad \frac{\partial^2 z}{\partial x \partial y} = 2y$ $\quad \frac{\partial^2 z}{\partial y^2} = 2x+2$

$P(0,0)$ $\quad \frac{\partial^2 z(P)}{\partial x^2} = 10 = A$ $\quad AC - B^2 = 2.10 - 0^2 > 0$ l. e. exist.

$\frac{\partial^2 z(P)}{\partial x \partial y} = 0 = B$ $\quad A = 10 > 0$ l. min.

$\frac{\partial^2 z(P)}{\partial y^2} = 2 = C$ $\quad (z(P) = 0)$

$R(-\frac{5}{3};0)$ ------ l. e. ex. $\quad$ l. max. $\quad z_{max}(R) = -2\frac{125}{27} + \frac{125}{9}$

$S(-1;2)$ ------ l. e. neex.

$D(-1;-2)$ ------ l. e. neex.

2

$z = e^{\frac{x}{2}}(x+y^2)$

$\frac{\partial z}{\partial x} = \frac{1}{2}e^{\frac{x}{2}}(x+y^2) + e^{\frac{x}{2}} = e^{\frac{x}{2}}(1 + \frac{x}{2} + \frac{y^2}{2}) = 0$ $\quad \left.\begin{array}{r} x+y^2+2 = 0 \\ 2y = 0 \end{array}\right\} M(-2;0)$

$\frac{\partial z}{\partial y} = 2ye^{\frac{x}{2}} = 0$

$\frac{\partial^2 z}{\partial x} = \frac{1}{2}e^{\frac{x}{2}}(1 + \frac{x}{2} + \frac{y^2}{2}) + \frac{e^{\frac{x}{2}}}{2} = \frac{e^{\frac{x}{2}}}{2}(\frac{2}{2} + \frac{x}{2} + \frac{y^2}{2})$ $\quad A = \frac{1}{2e}(2-1) = \frac{1}{2e}$

$\frac{\partial^2 z}{\partial x \partial y} = ye^{\frac{x}{2}}$ $\quad B = 0$

$\frac{\partial^2 z}{\partial y^2} = 2e^{\frac{x}{2}}$ $\quad AC - B^2 = \frac{1}{2e}\cdot\frac{2}{e} - 0 > 0$ $\quad C = \frac{2}{e}$

l. e. ex.

$A > 0$ l. min.

$z_{min}(M) = e^{-1}(-2+0) = -2.e^{-1}$

§ 13. Viazané extrémy f. 2 premenných.

$z = f(x;y)$ def. $M \subset E_2$

Nech prem. $x;y$ sú viazané rovnicou $g(x;y) = 0$ (1)

Ak chceme nájsť extrém $f(x;y)$ potom premenné $x;y$ musia vyhovovať aj rovnici (1); ktorú nazývame väzbou. Preto hovoríme o viazanom extréme. Môžeme poč. 2-akým spôsobom.

1.) Ak z (1) sa dá vyp. $y = \varphi(x)$ (2) potom (2) dosadíme do f. $f(x;y) \rightarrow z = f[x;\varphi(x)] \rightarrow z = F(x)$

Ďalej postupujeme ako v 1. sem.

→

Pr: $z = \frac{1}{x} + \frac{1}{y}$ nájsť extrém.

$x + y = 2$

$z = \frac{1}{x} + \frac{1}{2-x}$

$\frac{dz}{dx} = -\frac{1}{x^2} + \frac{1}{(2-x)^2}$

$\frac{dz}{dx} = 0 \qquad -\frac{1}{x^2} + \frac{1}{(2-x)^2} = 0 \qquad x \neq 2$

$\frac{-(2-x)^2 + x^2}{x^2(2-x)^2} = 0$

$x^2 - x^2 + 4x - 4 = 0$

$x = 1$

$\frac{\partial^2 z(1)}{dx^2} > 0 \sim$ l. min

$z_m(1) = \frac{1}{1} + \frac{1}{2-1} = 2.$

2. postup: Ak y sa nedá explicitne vyjadriť z rovnice (1) potom vytvoríme novú f. $F(x;y) = f(x;y) + \lambda g(x;y) \sim$ Lagrangeová f. Ďalej predpokl., že $F(x;y)$ splňuje tie isté podmienky pre lok. extrémy, ktoré boli uvedené pre neviazaný extrém.

Potom nutná podmienka je: $\frac{\partial F(x;y)}{\partial x} = 0$; $\frac{\partial F(x;y)}{\partial y} = 0$

$$\left.\begin{aligned} \frac{\partial f(x;y)}{\partial x} + \lambda \frac{\partial g(x;y)}{\partial x} &= 0 \\ \frac{\partial f(x;y)}{\partial y} + \lambda \frac{\partial g(x;y)}{\partial y} &= 0 \\ g(x;y) &= 0 \end{aligned}\right\}$$

rieš. týchto rovníc dostaneme stac. body a hodnoty λ.

$A = \frac{\partial^2 F(S)}{\partial x^2}$

$B = \frac{\partial^2 F(S)}{\partial x \partial y}$

$C = \frac{\partial^2 F(S)}{\partial y^2}$

S - stac. bod.

ďalej post. podobne ako v bode ...

Pr: $z = \frac{1}{x} + \frac{1}{y}$

$x + y = 2$

$F(x;y) = \frac{1}{x} + \frac{1}{y} + \lambda(x+y-2)$

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= -\frac{1}{x^2} + \lambda = 0 \\ \frac{\partial F}{\partial y} &= -\frac{1}{y^2} + \lambda = 0 \end{aligned}\right\}$$

$\frac{1}{x^2} = \frac{1}{y^2}$

$x^2 = y^2 \rightarrow x_{1,2} = \pm y$

1.) $x = +y$

2.) $x = -y$

$2y = 2$

$y = 1$

$T(1;1) \qquad \lambda = 1$

$\frac{\partial^2 F}{\partial x^2} = \frac{2}{x^3} \qquad A = \frac{2}{1} = 2$

$\frac{\partial^2 F}{\partial x \partial y} = 0 \qquad B = 0$

$\frac{\partial^2 F}{\partial y^2} = \frac{2}{y^3} \qquad C = \frac{2}{1} = 2$

ex. extr.

l. min.

$z_{min}(T) = \frac{1}{1} + \frac{1}{1} = 2$

2.) $x = -y$

$-y + y \neq 2 \sim$ nemá rieš.

14. V. 1969.

§ 14. n - rozmerný euklidovský priestor

Def. 1., : Sústava n reáln. č. : $a_1, a_2; a_3 \ldots a_n$ v tomto poradí sa nazýva bodom n-rozm. priestoru : $A(a_1; a_2, \ldots a_n)$

2., 2 body $A(a_1 \cdots a_n)$; $B(b_1 \ldots b_n)$ n-rozm. priestoru sa rovnajú vtedy a len vtedy ak $a_1 = b_1 \ldots\ldots a_n = b_n$.

3., Mn. vš. bodov n - rozm. priestoru sa naz. Euklid. n-r. priestorom ; E_n ak je v ňom definovaný rozdiel. 2 bodov $\rho(A,B)$ ktorá vyhovuje týmto podmienkam

a., $\rho(A;B) = \sqrt{(a_1-b_1)^2+(a_2-b_2)^2+\ldots+(a_n-b_n)^2}$

$\rho(A;B) = \rho(B;A)$

b., $\rho(A;B) \geq 0$; $\rho(A;B) = 0 \Leftrightarrow A = B$

c., $\rho(A;B) + \rho(B;C) \geq \rho(A;C)$ ~ trojuholníková vlastnosť.

Majme $P_0 \in E_n$; $P_0({}^0x_1; {}^0x_2 \ldots {}^0x_n)$. Pod okolím bodu P_0 rozumieme splnenie týchto nerovností : $|x_1 - {}^0x_1| < c$

$\ldots\ldots$

c ~ ľubov. pevne zvol. číslo. $|x_n - {}^0x_n| < c$

Pozn.: pojem spojitosti, limity a parc. derivácií sme def. v predch. paragrafoch.

Dif. f. n - premenných :

$z = f(x_1; x_2; \ldots x_n)$; predp. že f. má v O_{P_0} spoj. parc. der. 1. rádu ; potom :

$$dz = \frac{\partial f(P_0)}{\partial x_1} dx_1 + \ldots\ldots + \frac{\partial f(P_0)}{\partial x_n} dx_n .$$

$$dz = \sum_{i=1}^{n} \frac{\partial f(P_0)}{\partial x_i} dx_i$$

Pr.: $z = 3x^2yt - 2xt + ut^2 + 2xu^2$

$$dz = (6xyt - 2t + 2u^2)\,dx + (3x^2t)\,dy + (t^2 + 4ux)\,du + (3x^2y - 2x + 2ut)\,dt .$$

$P_0(1; 2; 3; 4)$

$dx = dy = du = dt$

$$dz_{(P_0)} = (48 - 8 + 18) \cdot 0{,}1 + \ldots$$

Funkcie n- premenných, dané implicitne

1.) $y = f(x)$ $\qquad F(x;y) = 0$

2.) $z = f(x;y)$ $\qquad F(x;y;z) = 0$

3.) $z = f(x_1; \dots x_n)$ $\qquad F(x_1; x_2; \dots x_n; z) = 0$

Derivácia: 1.) $$y' = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} \quad \Bigg| \quad \frac{\partial F}{\partial x}\cdot\underbrace{\frac{dx}{dx}}_{1} + \frac{\partial F}{\partial y}\cdot\underbrace{\frac{\partial y}{\partial x}}_{y'} = 0 \Rightarrow y' = \dots$$

2.) $F(x;y;z) = 0 \qquad z = f(x;y)$

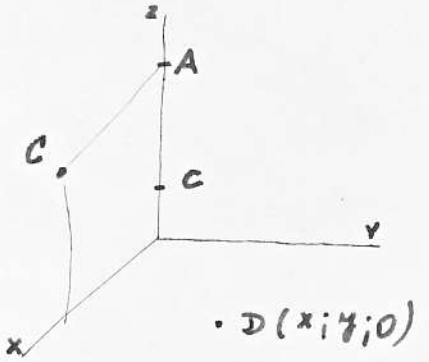
$$\frac{\partial F}{\partial x}\cdot\underbrace{\frac{dx}{dx}}_{1} + \frac{\partial F}{\partial y}\cdot\underbrace{\frac{dy}{dx}}_{0} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x} = 0 \longrightarrow$$

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$

analogicky pre $y; z$: ....

3.) $$\frac{\partial z}{\partial x_1} = -\frac{\frac{\partial F}{\partial x_1}}{\frac{\partial F}{\partial z}}; \quad \dots\dots \quad \frac{\partial z}{\partial x_n} = -\frac{\frac{\partial F}{\partial x_n}}{\frac{\partial F}{\partial z}}; \qquad \frac{\partial F}{\partial z} \neq 0$$

Pr. 1.) Dané sú 3 body roviny: A(0;0;12) B(0;0;4) C(8;0;8)
V rovine xy nájd. bod D tak, aby guľa prech. ABCD mala najmenší polomer.



$$r^2 = (x-m)^2+(y-n)^2+(z-p)^2$$
$$r^2 = m^2+n^2+(12-p)^2$$
$$r^2 = m^2+n^2+(4-p)^2$$
$$r^2 = (8-m)^2+n^2+(8-p)^2$$
$$r^2 = (x-m)^2+(y-n)^2+p^2$$

$p = 8$

$m = 3$

$$r^2 = (x-3)^2+(y-n)^2+64$$

$$\left.\begin{aligned}\frac{\partial r}{\partial x} &= \frac{x-3}{\sqrt{r^2}} = 0 \rightarrow x = 3\\ \frac{\partial r}{\partial y} &= \frac{y-n}{\sqrt{r^2}} = 0 \rightarrow y = n\end{aligned}\right\} S(3;n)$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{\sqrt{r} - (x-3)\cdot r^2/2\sqrt{r}}{r^2} = A = \frac{\sqrt{8}}{64}$$
$$\frac{\partial^2 r}{\partial y \partial x} = (x-3)\cdot\frac{\partial r}{\partial y}/2\sqrt{r^2} = B = \frac{0}{1}$$
$$\frac{\partial^2 r}{\partial y^2} = \frac{\sqrt{r} - (y-n)\frac{\partial r}{\partial y}\cdot/2\sqrt{r}}{r^2} = C = \frac{\sqrt{8}}{64}$$

$AC - B^2 > 0$ ~ ex. ex.

$A > 0$ ~ lok. min.

$$r = \sqrt{0^2+0^2+64} = 8$$

$$\Downarrow$$

$D(3;\sqrt{39};0)$ alebo $(3;-\sqrt{39};0)$

P.: 2 Na rov. $x+y-2z=0$ nájd. bod, ktorého súčet štvorcov vzd. od rovín $x+3z=6$ $y+3z=2$ je minimálny.

$$d=\frac{|Ax_0+By_0+Cz_0|}{\sqrt{A^2+B^2+C^2}} \qquad f=d_1^2+d_2^2 \qquad M\left(x;y;\frac{x+y}{2}\right) \text{ v rov } x+y-2z=0$$

$$d_1=\frac{\left|x+3\frac{x+y}{2}-6\right|}{\sqrt{1+9}}$$

$$d_2=\frac{\left|y+3\frac{x+y}{2}-2\right|}{\sqrt{1+9}}$$

$$f(x;y)=\frac{1}{10}\left[x-\frac{3}{2}(x+y)-6\right]^2+\frac{1}{10}\left[y+\frac{3}{2}(x+y)-2\right]^2=\dots$$

$$=\frac{1}{20}\left[17x^2+17y^2+30xy-72x-56y+80\right]$$

$$20f(x;y)=z=17x^2+17y^2+30xy+80-72x-56y$$

$$\left.\begin{aligned}\frac{\partial z}{\partial x}&=34x+30y-72\\ \frac{\partial z}{\partial y}&=34y+30x-56\end{aligned}\right\} \quad \left.\begin{aligned}34x+30y&=72\\ 34y+30x&=56\end{aligned}\right\} \Rightarrow S(3;-1)$$

$$\left.\begin{aligned}\frac{\partial^2 z}{\partial x^2}&=34=A\\ \frac{\partial^2 z}{\partial x\partial y}&=30=B\\ \frac{\partial^2 z}{\partial y^2}&=34=C\end{aligned}\right\} \quad \begin{aligned}&AC-B^2=34^2-30^2>0 \text{ extr. ex.}\\ &A>0 \text{ ost. min.}\end{aligned}$$

$$m=\frac{1}{20}\left[17\cdot 9+17\cdot 1+30\cdot 3(-1)+72\cdot 3-56(-1)+80\right]$$

$$\Downarrow$$

$$\underline{\underline{m=1.}}$$